PRACTICE EXAM 3

Printed Name: _____ **Class Start Time:** _____

- Record answers to Section 1 in the grid below, or you will receive a score of 0 for Section 1.
- Graphing Calculators without CAS are allowed. All other devices strictly prohibited.
- Only the first sheet of paper will be marked, and only returned if you come to office hours.

SECTION 1: Multiple Choice: Record One Clear Letter Answer Here (6 Points Each)

1	2	3	4	5	6	7	8	9	10

SECTION 2: Free Response (40 Points Total)

Show complete work. No credit for answers without justification.

Question 11 (15 Points): Interval of Convergence

Consider the following power series: $\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{5^n n}$

1. Find the radius of convergence.
2. Find the largest open interval of convergence.

Radius: _____ **Open Interval:** _____

Question 12 (10 Points): Arc Length
Find the length of the curve from 0 to π defined by $x = \cos(5t), y = \sin(5t)$

Exact answer: _____

Question 13 (15 Points): Taylor Series
For the function $f(x) = 2x^3 - x^2$:

1. Find the first **four non-zero** coefficients of the Taylor Series about $a = 2$.
2. Write down the Taylor polynomial approximation $P_3(x)$ about $a = 2$ using these coefficients.

Answer (a): _____

Answer (b): _____

SECTION 1: Multiple Choice Questions

Select the single best answer and write it in the grid on Page 1.

1. Given a series $\sum_{n=1}^{\infty} a_n$ where the Ratio Test yields a limit of $L = 0.1$, determine the correct conclusion regarding the convergence of the series.

A. No conclusion can be drawn

B. The series converges conditionally

C. The series converges absolutely

D. The series diverges

E. The series converges to 0.1

F. The series converges to 0

2. Find the power series for $f(x) = \dfrac{5x^4}{3+x}$ $(|x| < 3)$ using $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

A. $\sum_{n=0}^{\infty}(-1)^n \dfrac{5x^{n+4}}{3^{n+1}}$

B. $\sum_{n=0}^{\infty} \dfrac{5x^{n+4}}{3^n}$

C. $\sum_{n=0}^{\infty}(-1)^n \dfrac{x^{n+4}}{3^{n+1}}$

D. $\sum_{n=0}^{\infty}(-1)^n \dfrac{5x^{n+1}}{3^{n+1}}$

E. $\sum_{n=0}^{\infty} \dfrac{5x^{n+4}}{3^{n+1}}$

F. $\sum_{n=0}^{\infty}(-1)^n \dfrac{5x^{n+4}}{3^n}$

3. Find the largest open interval and radius of convergence I and R for the series: $\sum_{n=1}^{\infty} n!(x-4)^n 6^n$

A. $I = (-\infty, \infty), \quad R = \infty$

B. $I = (3, 5), \quad R = 1$

C. $I = (-4, 4), \quad R = 4$

D. $I = (3.5, 4.5), \quad R = 0.5$

E. $I = \{4\}, \quad R = 0$

F. $I = (-6, 6), \quad R = 6$

4. Given $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$, obtain the Maclaurin series for $f(x) = 5x\cos x$.

A. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 5x^{2n+1}}{(2n)!}$

B. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 5x^{2n+2}}{(2n)!}$

C. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 5x^{2n+1}}{(2n+1)!}$

D. $\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{5(2n)!}$

E. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 5x^{2n}}{(2n)!}$

F. $\sum_{n=0}^{\infty} \dfrac{(-1)^n 5x^{2n+2}}{(2n+1)!}$

5. Find the first three non-zero terms of the Maclaurin series for $f(x)$ on $(-1, 1)$, given that f satisfies: $f(0) = 0; \quad f'(0) = 2; \quad f''(0) = 0; \quad f'''(0) = -2; \quad f^{(4)}(0) = 0; \quad f^{(5)}(0) = 2$

A. $2x + \dfrac{2}{3!}x^3 + \dfrac{2}{5!}x^5$

B. $2x - 2x^3 + 2x^5$

C. $x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5$

D. $2 - \dfrac{2}{2!}x^2 + \dfrac{2}{4!}x^4$

E. $2x - \dfrac{2}{3!}x^3 + \dfrac{2}{5!}x^5$

F. $2x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5$

6. Find a parameterization below that is for a circle of radius 6 and center at $(-1, 2)$.

A. $x = -1 + 6\cos t,\ y = 2 + 6\sin(6t);\ 0 \le t \le 2\pi$

B. $x = -1 + 6\cos t,\ y = 2 + 6\sin t;\ 0 \le t \le 2\pi$

C. $x = -1 - 6\cos t,\ y = 2 - \sin t;\ 0 \le t \le 2\pi$

D. $x = -1 - 6\cos t,\ y = 2 - 6\sin(6t);\ 0 \le t \le 2\pi$

E. $x = 6\cos t,\ y = 6\sin t;\ 0 \le t \le 2\pi$

F. $x = 1 + 6\cos t,\ y = -2 + 6\sin t;\ 0 \le t \le 2\pi$

7. Find the first three non-zero terms of the Taylor series for $f(x) = \ln(x - 3)$ at $a = 4$.

A. $(x-4) - (x-4)^2 + 2(x-4)^3$

B. $(x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{6}(x-4)^3$

C. $(x-4) - \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3$

D. $(x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3$

E. $1 + (x-4) + \frac{1}{2}(x-4)^2$

F. $(x-4) + \frac{1}{2}(x-4)^2 + \frac{1}{3}(x-4)^3$

8. Find $\frac{dy}{dx}$ for the parametric curve $x = \cos(5t) - 3t^3;\quad y = t^2 - 4t$

A. $\dfrac{-\sin(5t) - 9t^2}{2t - 4}$

B. $\dfrac{2t - 4}{\cos(5t) - 3t^3}$

C. $\dfrac{t^2 - 4t}{5\sin(5t) - 9t^2}$

D. $\dfrac{2t - 4}{-5\sin(5t) - 9t^2}$

E. $\dfrac{\cos(5t) - 3t^3}{t^2 - 4t}$

F. $\dfrac{2t - 4}{-5\sin(5t) + 9t^2}$

9. Which simplified expression represents the length of the curve defined by $\begin{cases} x = \cos(5t) \\ y = \sin(7t) \end{cases}$ for $0 \le t \le \pi$?

A. $\displaystyle\int_0^\pi \sqrt{\cos^2(5t) + \sin^2(7t)}\, dt$

B. $\displaystyle\int_0^\pi \sqrt{-25\sin^2(5t) + 49\cos^2(7t)}\, dt$

C. $\displaystyle\int_0^\pi \sqrt{-5\sin(5t) + 7\cos(7t)}\, dt$

D. $\displaystyle\int_0^\pi \sqrt{25\sin^2(5t) + 49\cos^2(7t)}\, dt$

E. $\displaystyle\int_0^\pi \sqrt{5\sin(5t) + 7\cos(7t)}\, dt$

F. $\displaystyle\int_0^\pi \sqrt{25\cos^2(5t) + 49\sin^2(7t)}\, dt$

10. Eliminate the parameter to find the Cartesian equation of the curve: $x = 2 + 3\cos t,\quad y = 5 + 3\sin t$

A. $(x-2)^2 + (y-5)^2 = 3$

B. $(x-2)^2 + (y-5)^2 = 9$

C. $(x+2)^2 + (y+5)^2 = 9$

D. $(x-2)^2 + (y-5)^2 = 1/9$

E. $(x-2) + (y-5) = 9$

F. $\dfrac{(x-2)^2}{3} + \dfrac{(y-5)^2}{3} = 1$

Solutions

Multiple Choice Answers

1	2	3	4	5	6	7	8	9	10
C	A	E	A	E	B	C	D	D	B

Question 11 (15 Points): Interval of Convergence

Consider the following power series: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{5^n n}$

Find (a) the radius of convergence and (b) the largest open interval of convergence.

(a) Apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x+3)^{n+1}}{5^{n+1}(n+1)} \cdot \frac{5^n n}{(-1)^n (x+3)^n}\right|$$
$$= \lim_{n\to\infty} \left|\frac{x+3}{5} \cdot \frac{n}{n+1}\right| = \frac{|x+3|}{5} = L$$

Note: we used $\lim_{n\to\infty} \frac{n}{n+1} = 1$

For absolute convergence, we need $L < 1$

$$\frac{|x+3|}{5} < 1 \implies |x+3| < 5$$

From this inequality, we now know the radius of convergence: $\boxed{R = 5}$

(b) Interval of Convergence:

$$|x+3| < 5 \implies -5 < x+3 < 5 \implies -8 < x < 2$$

Since the problem only asks for the open interval, we do not need to check endpoints.
Open Interval of convergence: $\boxed{(-8, 2)}$

Question 12 (10 Points): Arc Length

Find the length of the curve from 0 to π defined by $x = \cos(5t), y = \sin(5t)$.
Derivatives with respect to t: $x'(t) = -5\sin(5t)$ $y'(t) = 5\cos(5t)$

Apply the arc length formula $\boxed{L = \displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt}$

$$L = \int_0^{\pi} \sqrt{(-5\sin(5t))^2 + (5\cos(5t))^2}\, dt$$
$$= \int_0^{\pi} \sqrt{25\sin^2(5t) + 25\cos^2(5t)}\, dt$$
$$= \int_0^{\pi} \sqrt{25(\sin^2(5t) + \cos^2(5t))}\, dt$$
$$= \int_0^{\pi} \sqrt{25(1)}\, dt = \int_0^{\pi} 5\, dt$$
$$= [5t]_0^{\pi} = 5\pi$$

Exact answer: $\boxed{5\pi}$

Question 13 (15 Points): Taylor Series
For the function $f(x) = 2x^3 - x^2$:

1. Find the first **four non-zero** coefficients of the Taylor Series about $a = 2$.
2. Write down the Taylor polynomial approximation $P_3(x)$ about $a = 2$ using these coefficients.

First, evaluate derivatives at $a = 2$ to find $f^{(n)}(a)$:

$$\begin{aligned} f(x) &= 2x^3 - x^2 \implies f(2) = 2(8) - 4 = 12 \\ f'(x) &= 6x^2 - 2x \implies f'(2) = 6(4) - 4 = 20 \\ f''(x) &= 12x - 2 \implies f''(2) = 12(2) - 2 = 22 \\ f'''(x) &= 12 \implies f'''(2) = 12 \end{aligned}$$

(a) The Taylor coefficients $c_n = \frac{f^{(n)}(a)}{n!}$:

$$\begin{aligned} c_0 &= \frac{12}{0!} = \boxed{12} \\ c_1 &= \frac{20}{1!} = \boxed{20} \\ c_2 &= \frac{22}{2!} = \frac{22}{2} = \boxed{11} \\ c_3 &= \frac{12}{3!} = \frac{12}{6} = \boxed{2} \end{aligned}$$

(b) The Taylor polynomial:

$$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}$$

$$= \boxed{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Thus, the Taylor polynomial approximation for $f(x) = 2x^3 - x^2$ centered at $a = 2$ is:

$$\boxed{T_3(x) = 12 + 20(x-2) + 11(x-2)^2 + 2(x-2)^3}$$