MAT 573 - Second Course in Complex Analysis

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The Goal of this second course is to hit a few highlight topics in complex analysis not covered in MAT 572 (which is the prerequisite for this course). These topics will be presented in lecture. The other goal is for each participant to choose a topic related to complex analysis for further study and report. Suggestions are given below.

Course Grade will be determined by an oral and written report on the chosen project. There will be no graded homework, quizzes or exams.

Outline of Topics Covered in Lecture.

1. *The Riemann Mapping Theorem* - which states that any simply connected region in the plane which is not the entire plane is analytically equivalent to the unit disc
   (a) Basic Outline of the Proof
   (b) Normal Families (and the Arzela Ascoli Theorem)
   (c) Completion of the Proof
2. *Runge's Approximation Theorem* - states that analytic functions on a domain $\Omega \subset \mathbb{C}$ can be approximated uniformly on compact subsets of Ω by entire functions if and only if Ω has no "holes" (i.e. the complement of Ω is connected). The classic "pole-pushing" argument will be given for its proof. Extensions of this theorem will be discussed to handle the case of two domains $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ to answer the question of when analytic functions on Ω_1 can be approximated by functions analytic on Ω_2.
3. *Solving the inhomogeneous Cauchy-Riemann equations*: $\dfrac{\partial f}{\partial \bar{z}} = g$ on a domain $\Omega \subset \mathbb{C}$, where g is a given smooth function on Ω and f is

the sought-after solution. When $\Omega = \mathbb{C}$ and g has compact support, a nice neat convolution formula with the Cauchy kernel will do the trick. The more general case will involve piecing together the solutions in the case where the right side is compactly supported together with the use of Runge's approximation theorem (as mentioned above).

4. *Infinite Products.* How are they defined and what are their basic properties. This topic is somewhat analogous to infinite series (and uses the results of this topic from MAT 572).

5. *An Introduction to Several Complex Variables.* An overview of why the theory of analytic functions several complex variables is so different than the theory of analytic functions in one complex variable. As time permits, an introduction to complex manifolds.

List of Possible Project Topics. There are many projects you can do. Here are a few examples:

1. *The Schwarz Christoffel Transformation.* used in mapping a polygonal region to the unit disc.

2. *Gamma Function and other Special Functions.* This topic uses infinite products so you'll have to cover this material on your own in view of the fact that it will be the last topic covered in lecture.

3. *Non Euclidean Geometry.* In the complex plane, a rigid motion is defined as a function which is comprised of a rotation followed by a translation, i.e. $T(z) = e^{i\theta} z + a$ where $0 \le \theta < 2\pi$ and $a \in \mathbb{C}$. It is easy to see that any rigid motion preserves Euclidean distance, i.e. $|T(z) - T(w)| = |z - w|$ for all $z,\ w \in \mathbb{C}$. For the unit disc Δ, recall that its automorphism group (i.e. the group of all analytic, 1-1, onto maps of the disc to itself) is the collection of maps $T : \Delta \to \Delta$ where

$$T(z) = e^{i\theta} \frac{z - a}{1 - \bar{a} z} \qquad 0 \le \theta < 2\pi, \quad |a| < 1$$

   There is a non-Euclidean distance function, D, on the unit disc which is invariant under this autormorphism group, i.e. $D(T(z), T(w)) = D(z, w)$ for all $z,\ w \in \Delta$ and all T in the automorphism group of Δ. Derive the formula for D and some of its properties. How does this distance compare to the usual Euclidean distance function?

4. *Topics in Harmonic Functions.* In MAT 572, we discussed one approach to solving the Dirichlét problem on a domain Ω, i.e. $\Delta u = 0$ on Ω and where u has given prescribed values on the boundary of Ω. In the cases we handled, the boundary value function was piecewise constant (typically 1 on part of the boundary and 0 on the rest). If the boundary value data is more general, then another approach is needed. One approach is the Poisson formula which computes the solution u via a convolution of the given boundary data with the Poisson kernel for the domain Ω. For domains such as the disc or the upper half plane, the Poisson kernel can be explicitly computed using complex analysis techniques.

5. *The Fourier Transform* is one of the key tools used in partial differential equations. For a function $f \in L^2(\mathbb{R})$, the Fourier transform is defined as
$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$
It turns out that the Fourier transform has an inverse which equals its L^2 adjoint and this remarkable property is one of the reasons it is such a valuable tool. Some complex analysis is commonly used in the establishment of this result.

6. *The Prime Number Theorem.* - Caution: not for the faint of heart as this will take more work. Although it has been known for many centuries that there are an infinite number of primes, it was not until the 19th century before serious progress had been made toward understanding how the primes are distributed. A big step in this understanding is the Prime Number Theorem (proved by Hadammard and independently by Pousin in 1896), which states the following: for a positive integer x, let $\pi(x)$ be the number of primes which are less than or equal to x; then
$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$$
Loosely speaking, this result states that the proportion of primes which are less than or equal to x is close to the fraction $1/\ln x$ when x is large. The proof requires a lot more complex analysis than the algebra of numbers and is one of many excellent examples of how research in number theory uses tools from other fields.

References. There are many excellent references for the above material. Here are just a few such books, all of which are carried by the ASU Library

1. Text for MAT 572 - Complex Analysis, 3rd edition, by J. Bak and D. Newman, Springer 2010 (project topics 1, 2, 4, 6)

2. Function Theory of One Complex Variable, by R. Greene and S. Krantz, Wiley, 1997 (project topics 2, 3, 4, 6)

3. Invitation to Complex Analysis, by R. P. Boas, Random House, 1987 (project topics, 2, 3, 4)

4. Introduction to Fourier analysis on Euclidean spaces, by Elias M. Stein and Guido Weiss, Princeton University Press, 1971. (project topics 4, 5)

5. A first course in wavelets with Fourier analysis, Albert Boggess and Francis J. Narcowich. Prentice Hall, 2001. - a second edition was published by Wiley, however this edition is not in the library; (project topics 4, 5)