School of Mathematical and Statistical Sciences
Arizona State University

Student: Brian Sweeney

Course: APM 503

Program: Mathematics MA

Instructor: Horst Thieme

Date: Fall 2019

APM 503 Project: Proofs Involving Inner Product Spaces
Brian Sweeney

A.1.1. (An inner product is uniquely determined by the norm) Let X be a vector space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $||\cdot||$.

(a) Show that $\langle u, v\rangle + \langle v, u\rangle = \frac{1}{2}(||u+v||^2 - ||u-v||^2)$.

Proof. Since $\mathbb{O} \in X$, let $u, v \in X$. Since $||\cdot||$ on X is induced by an inner product, $||u||^2 = \langle u, u\rangle$ for $u \in X$.

$$\begin{aligned}
\text{Then, } & \frac{1}{2}(||u+v||^2 - ||u-v||^2) \\
&= \frac{1}{2}(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle) \\
&= \frac{1}{2}(\langle u, u\rangle + \langle v, u\rangle + \langle v, u\rangle + \langle v, v\rangle - \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, v\rangle) \\
&= \frac{1}{2}(2\langle v, u\rangle + 2\langle u, v\rangle) \\
&= \langle u, v\rangle + \langle v, u\rangle
\end{aligned}$$

Thus, $\langle u, v\rangle + \langle v, u\rangle = \frac{1}{2}(||u+v||^2 - ||u-v||^2)$.

(b) Show that in a real inner product space $\langle u, v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2)$.

Proof. Since $\mathbb{O} \in X$, let $u, v \in X$. Since $||\cdot||$ on X is induced by an inner product, $||u||^2 = \langle u, u\rangle$ for $u \in X$. From the above proof (part (a)), we know that $\langle u, v\rangle + \langle v, u\rangle = \frac{1}{2}(||u+v||^2 - ||u-v||^2)$.

$$\begin{aligned}
\text{So, } \frac{1}{4}(||u+v||^2 - ||u-v||^2) &= \frac{1}{2}(\frac{1}{2}(||u+v||^2 - ||u-v||^2)) \\
&= \frac{1}{2}(\langle u, v\rangle + \langle v, u\rangle).
\end{aligned}$$

Since this is a real inner product space, $\langle u, v\rangle = \langle v, u\rangle$, so

$$\frac{1}{2}(\langle u, v\rangle + \langle v, u\rangle) = \frac{1}{2}(\langle u, v\rangle + \langle u, v\rangle) = \langle u, v\rangle$$

Thus, $\frac{1}{4}(||u+v||^2 - ||u-v||^2) = \langle u, v\rangle$.

(c) Show that, if X is a complex inner product space,

$$\langle u, v\rangle - \langle v, u\rangle = \frac{i}{2}(||u+iv||^2 - ||u-iv||^2)$$

and

$$\langle u, v\rangle = \frac{1}{4}(||u+v||^2 - ||u-v||^2 + i||u+iv||^2 - i||u-iv||^2).$$

Proof. Since $\mathbb{O} \in X$, let $u, v \in X$.

$$\begin{aligned}
\text{Then, } &\frac{i}{2}(||u+iv||^2 - ||u-iv||^2) \\
&= \frac{i}{2}(\langle u+iv, u+iv\rangle - \langle u-iv, u-iv\rangle) \\
&= \frac{i}{2}(\langle u, u\rangle + \langle iv, u\rangle + \langle u, iv\rangle + \langle iv, iv\rangle - \langle u, u\rangle + \langle iv, u\rangle + \langle u, iv\rangle - \langle iv, iv\rangle) \\
&= \frac{i}{2}(2\langle iv, u\rangle + 2\langle u, iv\rangle) \\
&= i\langle iv, u\rangle + i\langle u, iv\rangle \\
&= i^2\langle v, u\rangle + \langle iu, iv\rangle \\
&= \langle u, v\rangle - \langle v, u\rangle.
\end{aligned}$$

Thus, $\langle u, v\rangle - \langle v, u\rangle = \frac{i}{2}(||u+iv||^2 - ||u-iv||^2)$.

Since $\mathbb{O} \in X$, let $u, v \in X$. Then,

$$\frac{1}{4}(||u+v||^2 - ||u-v||^2 + i||u+iv||^2 - i||u-iv||^2) = \frac{1}{4}(||u+v||^2 - ||u-v||^2) + \frac{i}{4}(||u+iv||^2 - i||u-iv||^2)$$

. Using the result from (a), that $\langle u, v\rangle + \langle v, u\rangle = \frac{1}{2}(||u+v||^2 - ||u-v||^2)$, and the result above, that $\langle u, v\rangle - \langle v, u\rangle = \frac{i}{2}(||u+iv||^2 - ||u-iv||^2)$, we have the following:

$$\begin{aligned}
&\frac{1}{4}(||u+v||^2 - ||u-v||^2) + \frac{i}{4}(||u+iv||^2 - i||u-iv||^2) \\
&= \frac{1}{2}(\langle u, v\rangle + \langle v, u\rangle) + \frac{1}{2}(\langle u, v\rangle - \langle v, u\rangle) \\
&= \langle u, v\rangle.
\end{aligned}$$

Thus, $\frac{1}{4}(||u+v||^2 - ||u-v||^2 + i||u+iv||^2 - i||u-iv||^2) = \langle u, v\rangle$.

A.1.2. A real n x n matrix $A = (\alpha_{ij})$ is called symmetric if $\alpha_{ij} = \alpha_{ji}$ for all $i, j = 1, ..., n$.

(a) Show that a real n x n matrix A is symmetric if and only if $x \cdot (Ay) = (Ax) \cdot y$ for all $x, y \in \mathbb{R}^n$.

Proof.

($\rightarrow$) Suppose that a real n x n matrix A is symmetric. So, $\alpha_{ij} = \alpha_{ji}$ for all $i, j = 1, ..., n$. Let $x, y \in \mathbb{R}^n$; $x = (x_i)$ and $y = (y_i)$, with $i = 1, ..., n$.

Then, $Ay = (b_i)$, where $b_i = \sum_{j=1}^n \alpha_{ij} y_j$ for $i = 1, ..., n$. Thus,

$$\begin{aligned} x \cdot (Ay) &= \sum_{i=1}^n x_i b_i \\ &= \sum_{i=1}^n x_i (\sum_{j=1}^n \alpha_{ij} y_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n x_i \alpha_{ij} y_j. \end{aligned}$$

Now, we can change the order of the summations and factor out y_j:

$$\begin{aligned} x \cdot (Ay) &= \sum_{i=1}^n \sum_{j=1}^n x_i \alpha_{ij} y_j \\ &= \sum_{j=1}^n \sum_{i=1}^n x_i \alpha_{ij} y_j \\ &= \sum_{j=1}^n y_j \sum_{i=1}^n x_i \alpha_{ij}. \end{aligned}$$

Since $\alpha_{ij} = \alpha_{ji}$ for all $i, j = 1, ..., n$,

$$\begin{aligned} \sum_{j=1}^n y_j \sum_{i=1}^n x_i \alpha_{ij} &= \sum_{j=1}^n y_j \sum_{i=1}^n x_i \alpha_{ji} \\ &= y \cdot (Ax) \\ &= (Ax) \cdot y \end{aligned}$$

where the last equality follows from $\mathbb{R}^n$ being a real inner product space. Since $x, y \in \mathbb{R}^n$ were arbitrary, $x \cdot (Ay) = (Ax) \cdot y$ for all $x, y \in \mathbb{R}^n$.

($\leftarrow$) Suppose $x \cdot (Ay) = (Ax) \cdot y$ for all $x, y \in \mathbb{R}^n$. Denote $A = (\alpha_{ij})$, and let $x = (x_i)$

and $y = (y_i)$ be vectors in $\mathbb{R}^n$. Then, $Ay = (b_i)$ where $b_i = \sum_{j=1}^{n} \alpha_{ij} y_j$ for $i = 1, ..., n$, so

$$\begin{aligned} x \cdot (Ay) &= \sum_{i=1}^{n} x_i b_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \alpha_{ij} y_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i y_j. \end{aligned}$$

Also, $Ax = (c_i)$, where $c_i = \sum_{j=1}^{n} \alpha_{ij} x_j$ for $i = 1, ..., n$ and

$$\begin{aligned} (Ax) \cdot y &= \sum_{i=1}^{n} c_i y_i \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_j y_i. \end{aligned}$$

Changing the indices on this second inner product to match $x \cdot (Ay)$, gives that

$$\begin{aligned} (Ax) \cdot y &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_j y_i \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ji} x_i y_j. \end{aligned}$$

So, since $x \cdot (Ay) = (Ax) \cdot y$, we have that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i y_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \alpha_{ji} x_i y_j \tag{1}$$

for all $x, y \in \mathbb{R}^n$. Let $a, b \in \{1, ..., n\}$ and pick $x, y \in \mathbb{R}^n$, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, such that $x_a = 1$ and $x_i = 0$ for $i \neq a$, and $y_b = 1$ and $y_j = 0$ for $j \neq b$. Then, this equality gives us that $\alpha_{ab} = \alpha_{ba}$. Since (1) holds for all $x, y \in \mathbb{R}^n$, $\alpha_{ab} = \alpha_{ba}$ holds for all $a, b \in \{1, ..., n\}$ by picking $x, y \in \mathbb{R}^n$ in a similar manner. Thus, $\alpha_{ij} = \alpha_{ji}$ for all $i, j \in \{1, ..., n\}$, so A is symmetric.

(b) A symmetric matrix A is called positive definite if $x \cdot (Ax) > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. Show: A function $\langle , \rangle$ from $\mathbb{R}^n$ x $\mathbb{R}^n$ into $\mathbb{R}$ is an inner product on $\mathbb{R}^n$ if and only if there exists a positive definite symmetric matrix A such that $\langle x, y \rangle = x \cdot (Ay)$ for all $x, y \in \mathbb{R}^n$.

Proof.

($\rightarrow$) Suppose $\langle , \rangle$ from $\mathbb{R}^n$ x $\mathbb{R}^n$ into $\mathbb{R}$ is an inner product on $\mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$, where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Then, $x = x_1e_1 + ... + x_ne_n = \sum_{i=1}^n x_ie_i$ and $y = y_1e_1 + ... + y_ne_n = \sum_{i=1}^n y_ie_i$, where $e_1, ..., e_n$ denote the standard basis of $\mathbb{R}^n$. By the distributive law, we have that

$$\begin{aligned}\langle x, y\rangle &= \langle \sum_{i=1}^n x_ie_i, \sum_{j=1}^n y_je_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle x_ie_i, y_je_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n x_iy_j\langle e_i, e_j\rangle.\end{aligned}$$

Pick A to be a matrix such that $\alpha_{ij} = \langle e_i, e_j\rangle$ for $i, j = 1, ..., n$.

Then, $\langle x, y\rangle = \sum_{i=1}^n \sum_{j=1}^n x_iy_j\alpha_{ij} = x \cdot (Ay)$. Now, we must show that A is symmetric and positive definite.

By the symmetry of the inner product, $\langle x, y\rangle = \langle y, x\rangle$, so $x \cdot (Ay) = \langle x, y\rangle = \langle y, x\rangle = y \cdot (Ax)$. By part (a), A is then a symmetric matrix. By the positivity of $\langle , \rangle$, we have that $\langle u, u\rangle > 0$ for all $u \in \mathbb{R}^n$, $u \neq \mathbb{O}$, so A is positive definite. Therefore, A is a positive definite matrix that satisfies $\langle x, y\rangle = x \cdot (Ay)$ for all $x, y \in \mathbb{R}^n$.

($\leftarrow$) Suppose there exists a positive definite matrix A such that $\langle x, y\rangle = x \cdot (Ay)$ for all $x, y \in \mathbb{R}^n$. Let $x, y, z \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$.

(i) Since A is symmetric, $x \cdot (Ay) = (Ax) \cdot y$ by part (a). Also, since the Euclidean inner product is an inner product on $\mathbb{R}^n$, $\cdot$ is symmetric so $(Ax) \cdot y = y \cdot (Ax)$. Thus, $\langle x, y\rangle = x \cdot (Ay) = y \cdot (Ax) = \langle y, x\rangle$, so $\langle , \rangle$ is symmetric.

(ii) Since the Euclidean inner product is an inner product on $\mathbb{R}^n$, $(\alpha x) \cdot (Ay) = \alpha[x \cdot (Ay)]$. So, $\langle \alpha x, y\rangle = (\alpha x) \cdot (Ay) = \alpha[x \cdot (Ay)] = \alpha\langle x, y\rangle$, so the associate law holds.

(iii) Again, since $\cdot$ is an inner product on $\mathbb{R}^n$, we have $\langle x + y, z\rangle = (x + y) \cdot (Az) = [x \cdot (Az)] + [y \cdot (Az)] = \langle x, z\rangle + \langle y, z\rangle$. Thus the distributive law holds for $\langle , \rangle$.

(iv) Since A is a positive definite matrix, $u \cdot (Au) > 0$ for all $u \in \mathbb{R}^n$, $u \neq \mathbb{O}$. Let $x \in \mathbb{R}^n$, $x \neq \mathbb{O}$. Then, $\langle x, x\rangle = x \cdot (Ax) > 0$, so $\langle , \rangle$ is positive definite. Thus, $\langle , \rangle$ is an inner product on $\mathbb{R}^n$.

A.1.3. Let A be a positive definite symmetric n x n matrix and $\cdot$ denote the inner product on $\mathbb{R}^n$.
Show: $|x \cdot (Ay)|^2 \leq [x \cdot (Ax)][y \cdot (Ay)]$ for all $x, y \in \mathbb{R}^n$ with equality holding if and only if x and y are linearly dependent.

Proof.

Let A be a positive definite symmetric n x n matrix. To prove this inequality, we will first show that $\langle , \rangle$ defined by $\langle x, y \rangle = x \cdot (Ay)$ is an inner product on $\mathbb{R}^n$. Let $x, y, z \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

(i) We have that $\langle x, y \rangle = x \cdot (Ay) = y \cdot (Ax) = \langle y, x \rangle$ from the result of A.1.2(a) and the commutativity of the inner product on $\mathbb{R}$. So, $\langle x, y \rangle = \langle y, x \rangle$.

(ii) We can use the fact that $\cdot$ is an inner product on $\mathbb{R}^n$ to rewrite $\langle \alpha x, y \rangle$ as follows: $\langle \alpha x, y \rangle = (\alpha x) \cdot (Ay) = \alpha [x \cdot (Ay)] = \alpha \langle x, y \rangle$. So, $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(iii) Consider $\langle x + y, z \rangle = (x + y) \cdot (Az)$. Again, since $\cdot$ is an inner product on $\mathbb{R}^n$,

$$\begin{aligned} (x + y) \cdot (Az) &= [x \cdot (Az)] + [y \cdot (Az)] \\ &= \langle x, z \rangle + \langle y, z \rangle. \end{aligned}$$

Thus, $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(iv) Let $w \in \mathbb{R}^n$,$w \neq \mathbb{O}$. Since A is positive definite, $\langle w, w \rangle = w \cdot (Aw) > 0$ by definition. Thus, $\langle w, w \rangle > 0$ for $w \neq \mathbb{O}$.

Therefore, we have that $\langle , \rangle$ is an inner product on $\mathbb{R}^n$. Applying the Cauchy-Schwarz Inequality (Theorem A.2), we have that for all $x, y \in \mathbb{R}^n$,

$$\begin{gathered} |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \text{ or equivalently,} \\ |x \cdot (Ay)|^2 \leq [x \cdot (Ax)][y \cdot (Ay)], \end{gathered}$$

with equality if and only if x and y are linearly dependent.

A.1.4. Consider $\ell^2 = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}}; ||x||_2 < \infty\}$ where

$$||x||_2^2 = \sum_{n=1}^{\infty} |x_n|^2$$

.

Show:

(a) For each $x = (x_n)$ and $y = (y_n)$ in ℓ^2, the series

$$\sum_{k=1}^{\infty} x_k \bar{y}_k =: \langle x, y \rangle$$

converges in $\mathbb{C}$ (with absolute value) and defines an inner product on ℓ^2.

Proof.

We first show that ℓ^2 is a linear subspace of $\mathbb{C}^{\mathbb{N}}$. Let $x, y \in \ell^2$ and $\alpha \in \mathbb{C}$.

(i) Since $||x||_2 < \infty$, there exists $c \in \mathbb{R}$, $c > 0$ such that $||x||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2} < c$, so $||x||_2^2 = \sum_{n=1}^{\infty} |x_n|^2 < c^2$. So, for all $m \in \mathbb{N}$, $\sum_{n=1}^{m} |x_n|^2 < c^2$. Then, for all $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{m} |\alpha x_n|^2 &= \sum_{n=1}^{m} |\alpha|^2 \, |x_n|^2 \\ &= |\alpha|^2 \sum_{n=1}^{m} |x_n|^2 \\ &< |\alpha|^2 c^2.\end{aligned}$$

Since this is true for all $m \in \mathbb{N}$, $\sum_{n=1}^{\infty} |\alpha x_n|^2 < |\alpha|^2 c^2$, so $||\alpha x||_2 = \sqrt{\sum_{n=1}^{\infty} |\alpha x_n|^2} < \sqrt{|\alpha|^2 c^2} < \infty$, meaning that $\alpha x \in \ell^2$.

(ii) Since $||x||_2 < \infty$ and $||y||_2 < \infty$, there exists $c, d \in \mathbb{R}$, $c, d > 0$ such that $||x||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2} < c$ and $||y||_2 = \sqrt{\sum_{n=1}^{\infty} |y_n|^2} < d$, so $||x||_2^2 = \sum_{n=1}^{\infty} |x_n|^2 < c^2$ and $||y||_2^2 = \sum_{n=1}^{\infty} |y_n|^2 < d^2$. Then, for all $m \in \mathbb{N}$, $\sum_{n=1}^{m} |x_n|^2 < c^2$ and $\sum_{n=1}^{m} |y_n|^2 < d^2$. So, for all

$m \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{m} |x_n + y_n|^2 &= \sum_{n=1}^{m} |x_n + y_n|\ |x_n + y_n| \\ &\leq \sum_{n=1}^{m} |x_n|\ (|x_n + y_n|) + |y_n|\ (|x_n + y_n|) \\ &\leq \sum_{n=1}^{m} |x_n|\ (|x_n| + |y_n|) + |y_n|\ (|x_n| + |y_n|) \\ &= \sum_{n=1}^{m} |x_n|^2 + 2|x_n|\ |y_n| + |y_n|^2 \\ &< c^2 + 2cd + d^2\end{aligned}$$

Since this is true for all $m \in \mathbb{N}$, $\sum_{n=1}^{\infty} |x_n+y_n|^2 < c^2+2cd+d^2$, so $||x+y||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n + y_n|^2} < \sqrt{c^2 + 2cd + d^2} < \infty$, meaning that $x + y \in \ell^2$. Thus, ℓ^2 is a linear subspace of $\mathbb{C}^{\mathbb{N}}$.

Now we must show that the series

$$\sum_{k=1}^{\infty} x_k \bar{y}_k =: \langle x, y \rangle$$

converges in $\mathbb{C}$ for each $x = (x_n)$ and $y = (y_n)$. Let $x, y \in \ell^2$, so $x = (x_n)$ and $y = (y_n)$ since x, y are sequences. Also, $||x||_2 < \infty$ and $||y||_2 < \infty$ since $x, y \in \ell^2$. So, there exists $c, d \in \mathbb{R}$, $c, d > 0$ such that $||x||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2} < c$ and $||y||_2 = \sqrt{\sum_{n=1}^{\infty} |y_n|^2} < d$, and by extension, $||x||_2^2 = \sum_{n=1}^{\infty} |x_n|^2 < c^2$ and $||y||_2^2 = \sum_{n=1}^{\infty} |y_n|^2 < d^2$. So, for all $m \in \mathbb{N}$, $\sum_{n=1}^{m} |x_n|^2 < c^2$ and $\sum_{n=1}^{m} |y_n|^2 < d^2$. Then, by the triangle inequality, we have

$$\begin{aligned}|\sum_{k=1}^{m} x_k \bar{y}_k| &\leq \sum_{k=1}^{m} |x_k \bar{y}_k| \\ &= \sum_{k=1}^{m} |x_k|\ |\bar{y}_k| \\ &= \sum_{k=1}^{m} |x_k|\ |y_k|.\end{aligned}$$

Note that we can bound $\sum_{k=1}^{m} |x_k|\ |y_k|$ by looking at the relation between $|x_k|$ and $|y_k|$ for each $k = 1, ..., m$. If $|x_k| \leq |y_k|$ for a given $k = 1, ..., m$, then, $|x_k|\ |y_k| \leq |y_k|^2$. If $|x_k| > |y_k|$ for a given $k = 1, ..., m$, then, $|x_k|\ |y_k| < |x_k|^2$. So, combining these two, we get that $\sum_{k=1}^{m} |x_k|\ |y_k| \leq \sum_{k=1}^{m} (|x_k|^2 + |y_k|^2) < c^2 + d^2$ for all $m \in \mathbb{N}$. Thus, $|\sum_{k=1}^{m} x_k \bar{y}_k| \leq \sum_{k=1}^{m} |x_k \bar{y}_k| \leq c^2 + d^2$ for all $m \in \mathbb{N}$. So, the partial sums $|\sum_{k=1}^{m} x_k \bar{y}_k|$, which is a non-negative series, is bounded for all $m \in \mathbb{N}$. Thus $|\sum_{k=1}^{\infty} x_k \bar{y}_k| < c^2 + d^2$, meaning that

$|\sum_{k=1}^{\infty} x_k \bar{y}_k|$ converges in $\mathbb{C}$ with the absolute value by Theorem 2.38. Now, we must prove that $\langle , \rangle$ defines an inner product on the vector space ℓ^2.

(i) Let $x, y \in \ell^2$. Note that $\langle y, x \rangle$ exists since $x, y \in \mathbb{C}^{\mathbb{N}}$, so $\sum_{k=1}^{\infty} y_k \bar{x}_k$ converges. Then, we can apply the complex conjugation over the sum as follows:

$$\begin{aligned}\overline{\langle y, x \rangle} &= \overline{\langle y, x \rangle} \\ &= \overline{\sum_{k=1}^{\infty} y_k \bar{x_k}} \\ &= \sum_{k=1}^{\infty} \bar{y}_k x_k \\ &= \sum_{k=1}^{\infty} x_k \bar{y}_k \\ &= \langle x, y \rangle.\end{aligned}$$

Thus, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

(ii) Let $x, y \in \ell^2$ and $\alpha \in \mathbb{C}$. Since $\sum_{k=1}^{\infty} x_k \bar{y}_k =: \langle x, y \rangle$ converges, we have that

$$\begin{aligned}\alpha \langle x, y \rangle &= \alpha \sum_{k=1}^{\infty} x_k \bar{y_k} \\ &= \sum_{k=1}^{\infty} \alpha x_k \bar{y}_k \\ &= \langle \alpha x, y \rangle.\end{aligned}$$

Thus, the associative law holds.

(iii) Let $x, y, z \in \ell^2$. Since $\sum_{k=1}^{\infty} x_k \bar{z}_k =: \langle x, z \rangle$ converges and $\sum_{k=1}^{\infty} y_k \bar{z}_k =: \langle y, z \rangle$ converges, we have that

$$\begin{aligned}\langle x, z \rangle + \langle y, z \rangle &= \sum_{k=1}^{\infty} x_k \bar{z_k} + \sum_{k=1}^{\infty} y_k \bar{z_k} \\ &= \sum_{k=1}^{\infty} (x_k + y_k) \bar{z_k} \\ &= \langle x + y, z \rangle\end{aligned}$$

Thus, the distributive law holds for $\langle , \rangle$.

(iv) Let $x \in \ell^2$ such that $x = (x_n)$ is not the zero sequence. Then, $\langle x, x\rangle = \sum_{k=1}^{\infty} x_k \bar{x}_k$ since this series converges in $\mathbb{C}$ with the absolute value. Since $x = (x_n)$ is not the zero sequence, there exists some $j \in \mathbb{N}$ such that $x_j \neq 0$. So, $\langle x, x\rangle = \sum_{k=1}^{\infty} x_k \bar{x}_k \geq x_j \bar{x}_j > 0$. Thus, $\langle , \rangle$ is positive definite. Therefore, the series defines an inner product on ℓ^2.

(b) ℓ^2 with this inner product is a Hilbert space.

Proof.

Let $(x_n(k))_{k\in\mathbb{N}}$ be a Cauchy sequence in ℓ^2, with k being the index for the sequence x_n (So, x_n is a sequence in ℓ^2 and for a fixed n, $(x_n(k))_{k\in\mathbb{N}} \in \ell^2$). Let $\epsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $||x_n - x_m||_2 < \sqrt{\frac{\epsilon}{2}}$, so $||x_n - x_m||_2^2 < \frac{\epsilon}{2}$. We also have that $||x_n - x_m||_2^2 = \sum_{k=1}^{\infty} |x_n(k) - x_m(k)|^2$, so $\epsilon > ||x_n - x_m||_2^2 \geq |x_n(k) - x_m(k)|$ for any $k \in \mathbb{N}$ and $n, m \geq N$. This implies that (x_n) is a uniform Cauchy sequence on $\mathbb{N}$. By Remark 2.21, $(x_n(k))$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ with the absolute value is a complete metric space, for each $k \in \mathbb{N}$, there exists $z(k) \in \mathbb{C}$ such that $x_n(k) \to z(k)$ as $n \to \infty$.

Consider the sequence $z = (z(k))_{k\in\mathbb{N}}$. Let $j \in \mathbb{N}$. Since (x_n) is a Cauchy sequence, there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$, $||x_n - x_m||_2 < \frac{\epsilon}{2}$. So, $\sqrt{\sum_{k=1}^{j} |x_n(k) - x_m(k)|^2} \leq \sqrt{\sum_{k=1}^{\infty} |x_n(k) - x_m(k)|^2} < \frac{\epsilon}{2}$. Since $x_n(k) \to z(k)$ as $n \to \infty$ for each $k \in \mathbb{N}$, there exists some $m_{kj} \in \mathbb{N}$ such that $|x_n(k) - z(k)| < \frac{\epsilon}{2\sqrt{j}}$ for all $n \geq m_{kj}$. So, $\sum_{k=1}^{j} |x_n(k) - z(k)|^2 < \frac{\epsilon^2}{4}$, and $\sqrt{\sum_{k=1}^{j} |x_n(k) - z(k)|^2} < \frac{\epsilon}{2}$ for all $n \geq m_{kj}$. Let $m \geq \max\{m_{kj}, M\}$. Then for all $j \in \mathbb{N}$ and $n \geq M$, we have

$$\begin{aligned}
\sqrt{\sum_{k=1}^{j} |x_n(k) - z(k)|^2} &= \sqrt{\sum_{k=1}^{j} |x_n(k) - x_m(k) + x_m(k) - z(k)|^2} \\
&\leq \sqrt{\sum_{k=1}^{j} |x_n(k) - x_m(k)|^2} + \sqrt{\sum_{k=1}^{j} |x_m(k) - z(k)|^2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since this holds for all $j \in \mathbb{N}$, $\sqrt{\sum_{k=1}^{\infty} |x_n(k) - z(k)|^2} < \epsilon$, meaning that $||x_n - z||_2 = \sqrt{\sum_{k=1}^{\infty} |x_n(k) - z(k)|^2} < \epsilon$ for all $n \geq M$. Thus, since M does not depend on k, (x_n) converges uniformly to z.

Now, we must show that $z \in \ell^2$. Consider $\sum_{k=1}^{m} |z(k)|^2 = \sum_{k=1}^{m} |\lim_{n\to\infty} x_n(k)|^2$. Since $(x_n(k))_{k\in\mathbb{N}}$ is a Cauchy sequence in ℓ^2, $(x_n(k))_{k\in\mathbb{N}}$ is bounded in ℓ^2. So, there exists $c > 0$ such that for all $n \in \mathbb{N}$, $||x_n||_2 = \sqrt{\sum_{k=1}^{\infty} |x_n(k)|^2} < c$. Then, $||x_n||_2^2 = \sum_{k=1}^{\infty} |x_n(k)|^2 < c^2$. So, for all $m \in \mathbb{N}$ and $n \in \mathbb{N}$, $\sum_{k=1}^{m} |x_n(k)|^2 < c^2$. Then, by the definition of limit, for all $m \in \mathbb{N}$, $\sum_{k=1}^{m} |z(k)|^2 = \sum_{k=1}^{m} |\lim_{n\to\infty} x_n(k)|^2 \leq c^2$. Since this holds for all $m \in \mathbb{N}$, by

Theorem 2.38, we have that $\sum_{k=1}^{\infty} |z(k)|^2$ converges. So, $||z||_2^2 = \sum_{k=1}^{\infty} |z(k)|^2 < \infty$, meaning that $||z||_2 < \infty$. Thus, $z \in \ell^2$. Therefore, $x_n \to z$ as $n \to \infty$ and $z \in \ell^2$, so ℓ^2 is a Hilbert space.

A.1.5. Let X be an inner product space over $\mathbb{K}$ and (x_n), (y_n) be Cauchy sequences in X. Show: The sequence $(\langle x_n, y_n\rangle)$ converges in $\mathbb{K}$.

Proof.

Let $\epsilon > 0$. Since (x_n), (y_n) are Cauchy sequences in X, there exists $N, M \in \mathbb{N}$ such that $||x_n - x_m|| < \epsilon$ for all $n, m \geq \mathbb{N}$ and $||y_n - y_m|| < \epsilon$ for all $n, m \geq M$. So, we have that $||x_n - x_m|| \to 0$ as $n, m \to \infty$ and $||y_n - y_m|| \to 0$ as $n, m \to \infty$. Note that for all $n, m \in \mathbb{N}$,

$$\begin{aligned}\langle x_n, y_n\rangle - \langle x_m, y_m\rangle &= \langle x_n, y_n\rangle - \langle x_m, y_n\rangle + \langle x_m, y_n\rangle - \langle x_m, y_m\rangle \\ &= \langle x_n - x_m, y_n\rangle + \langle x_m, y_n - y_m\rangle.\end{aligned}$$

So, $|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| = |\langle x_n - x_m, y_n\rangle + \langle x_m, y_n - y_m\rangle|$. By the triangle inequality and the Cauchy-Schwarz ineqaulity,

$$\begin{aligned}|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| &= |\langle x_n - x_m, y_n\rangle + \langle x_m, y_n - y_m\rangle| \\ &\leq |\langle x_n - x_m, y_n\rangle| + |\langle x_m, y_n - y_m\rangle| \\ &\leq ||x_n - x_m||\ ||y_n|| + ||x_m||\ ||y_n - y_m||.\end{aligned}$$

Since $||x_n - x_m|| \to 0$ and $||y_n - y_m|| \to 0$ as $n, m \to \infty$, $||x_n - x_m||\ ||y_n|| + ||x_m||\ ||y_n - y_m|| \to 0$ as $n, m \to \infty$. Thus, $|\langle x_n, y_n\rangle - \langle x_m, y_m\rangle| \to 0$ as $n, m \to \infty$, so $(\langle x_n, y_n\rangle)$ is a Cauchy sequence in $\mathbb{K}$. Since $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, $\mathbb{K}$ with the absolute value is a complete space. Thus, since $(\langle x_n, y_n\rangle)$ is a Cauchy sequence in $\mathbb{K}$, $(\langle x_n, y_n\rangle)$ converges in $\mathbb{K}$.

A.1.6. Let X be an inner product space and x, y be points in X, $\alpha \in \mathbb{K}$, and (x_n), (y_n) be sequences in X and (α_n) a sequence in $\mathbb{K}$.

Show: If $x_n \to x$, $y_n \to y$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $\langle \alpha_n x_n, y_n \rangle \to \langle \alpha x, y \rangle$ as $n \to \infty$.

Proof.

Suppose $x_n \to x$, $y_n \to y$, and $\alpha_n \to \alpha$ as $n \to \infty$. Let $\epsilon > 0$. Since $x_n \to x$ as $n \to \infty$, each component of x_n, $x_n^j e_j$, where e_j is the jth canonical basis vector, converges to the same component in x, $x^j e_j$. So, for each j, $x_n^j \to x^j$ in $\mathbb{K}$ as $n \to \infty$. By the limit properties of $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, $\alpha_n x_n^j \to \alpha x^j$ as $n \to \infty$ for each j. So, $\alpha_n x_n \to \alpha x$ as $n \to \infty$. So, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $||\alpha_n x_n - \alpha x|| < \frac{\epsilon}{2||y||}$. Since $\alpha_n x_n \to \alpha x$ as $n \to \infty$, $\alpha_n x_n$ is bounded for all $n \in \mathbb{N}$. So, for all $n \in \mathbb{N}$, $||\alpha_n x_n|| \leq c$, for some $c > 0$. Since $y_n \to y$ as $n \to \infty$, there exists $M \in \mathbb{N}$ such that for all $n \geq \mathbb{M}$, $||y_n - y|| < \frac{\epsilon}{2c}$. Pick $L = \max\{M, N\}$. Then, for $n \geq L$,

$$
\begin{aligned}
&|\langle \alpha_n x_n, y_n \rangle - \langle \alpha x, y \rangle| \\
&= |\langle \alpha_n x_n, y_n \rangle - \langle \alpha_n x_n, y \rangle + \langle \alpha_n x_n, y \rangle| - \langle \alpha x, y \rangle| \\
&= |\langle \alpha_n x_n, y_n - y \rangle + \langle \alpha_n x_n - \alpha x, y \rangle| \\
&\leq ||\alpha_n x_n|| \; ||y_n - y|| + ||\alpha_n x_n - \alpha x|| \; ||y|| \text{ (Cauchy-Schwarz)} \\
&< ||\alpha_n x_n|| \; \frac{\epsilon}{2c} + \frac{\epsilon}{2||y||} \; ||y|| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

Thus, $\langle \alpha_n x_n, y_n \rangle \to \langle \alpha x, y \rangle$ as $n \to \infty$.

A.1.7. Let X be an inner product space, $x \in X$ and (x_n) a sequence in X. Show: $x_n \to x$ as $n \to \infty$ if and only if $||x_n|| \to ||x||$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$.

Proof.

($\rightarrow$) Suppose $x_n \to x$ as $n \to \infty$. Let $\epsilon > 0$. Then, there exists $N \in \mathbb{N}$ such that $||x_n - x|| < \epsilon$ for all $n \geq N$. By the reverse triangle inequality, $\epsilon > ||x_n - x|| \geq |\ ||x_n|| - ||x||\ |$ for all $n \geq N$. So, $||x_n|| \to ||x||$ as $n \to \infty$. Since $x_n \to x$ as $n \to \infty$, there exists $M \in \mathbb{N}$ such that $||x_n - x|| < \frac{\epsilon}{||x||}$ for all $n \geq M$. Then, $|\langle x_n, x \rangle - \langle x, x \rangle| = |\langle x_n - x, x \rangle| \leq ||x_n - x||\ ||x|| < \frac{\epsilon}{||x||}\ ||x|| = \epsilon$ by Cauchy-Schwarz. Thus, $||x_n|| \to ||x||$ as $n \to \infty$ as well.

($\leftarrow$) Suppose $||x_n|| \to ||x||$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$. Then, $\langle x_n, x \rangle \to \langle x, x \rangle = ||x||^2$ as $n \to \infty$.

$$
\begin{aligned}
\text{So, } ||x_n - x||^2 &= \langle x_n - x, x_n - x \rangle \\
&= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle \\
&= ||x_n||^2 - \langle x_n, x \rangle - \langle x, x_n \rangle + ||x||^2.
\end{aligned}
$$

Then, as $n \to \infty$, the above equation,

$$||x_n||^2 - \langle x_n, x \rangle - \langle x, x_n \rangle + ||x||^2 \to ||x||^2 - ||x||^2 - ||x||^2 + ||x||^2 = 0.$$

So, as $n \to \infty$, $||x_n - x||^2 \to 0$. Therefore, $||x_n - x|| \to 0$ as $n \to \infty$, meaning that $x_n \to x$ as $n \to \infty$.

A.1.8. Let X be an inner product space. Let $y \in X$ be fixed but arbitrary. Define $f, g : X \to \mathbb{C}$ by

$$f(x) = \langle x, y \rangle, \ g(x) = \langle y, x \rangle, \ x \in X$$

Then f and g are Lipschitz continuous with Lipschitz constant $||y||$.

Proof.

Let $x, z \in X$. Let d be the metric induced by the norm on X.

$$\begin{aligned} \text{Then, } |f(x) - f(z)| &= |\langle x, y \rangle - \langle z, y \rangle| \\ &= |\langle x - z, y \rangle| \\ &\leq ||x - z|| \ ||y|| \\ &= d(x, z) \ ||y|| \end{aligned}$$

by the Cauchy-Schwarz inequality. Thus, $|f(x) - f(z)| \leq ||y|| \ d(x, z)$, so f is Lipschitz continuous with Lipschitz constant $||y||$.

To show g is Lipschitz continuous, again let $x, z \in X$.

$$\begin{aligned} \text{Then, } |g(x) - g(z)| &= |\langle y, x \rangle - \langle y, z \rangle| \\ &= |\overline{\langle x, y \rangle} - \overline{\langle z, y \rangle}| \\ &= |\overline{\langle x, y \rangle - \langle z, y \rangle}| \end{aligned}$$

from the distributive law and properties of the complex conjugate.

$$\begin{aligned} \text{Then, } |\overline{\langle x, y \rangle - \langle z, y \rangle}| &= |\overline{\langle x - z, y \rangle}| \\ &= |\langle y, x - z \rangle| \end{aligned}$$

from the distribute law and the properties of an inner product.

Finally, $|\langle y, x-z \rangle| \leq ||y|| \ ||x-z|| = ||y|| \ d(x, z)$ by the Cauchy-Schwarz inequality. Thus, $|g(x) - g(z)| \leq ||y|| \ d(x, z)$. So, g is Lipschitz continuous with $||y||$ a Lipschitz constant.

A.1.9. Let M be a complete linear subspace of the inner product space X.

Show: Each vector $u \in X$ has a unique representation $u = v + w$ such that $v \in M$ and $\langle w, z \rangle = 0$ for all $z \in M$. (The vector $v \in M$ is called the orthogonal projection of u on M)

Proof.

By Remark 1.10, linear subspaces of a vector space are convex, so M is convex. Let $u \in X$. Then, by Proposition A.9, for each vector $u \in X$, there exists a unique $v \in M$ such that $d(u, M) = ||u - v||$. Let $w = u - v$. Now, we must show that $\langle w, z \rangle = 0$ for all $z \in M$. Let $z \in M$ with $||z|| = 1$ by normalizing u,v, and w to z. Let $\alpha \in \mathbb{K}$ and consider $\phi(\alpha) = ||w - \alpha z||^2$. Then, $||w - \alpha z||^2 = ||u - v - \alpha z||^2 = ||u - (v + \alpha z)||^2$. Since $v, z \in M$ and $\alpha \in \mathbb{K}$, $v + \alpha z \in M$ by definition of a linear subspace. Since $v \in M$ is the unique vector in M such that $||u - v|| = d(u, M) = \inf_{y \in M}\{||u - y||; y \in M\}$, $||u - (v + \alpha z)|| \geq ||u - v||$. So, the minimum of $\phi(\alpha) = ||w - \alpha z||^2 = ||u - (v + \alpha z)||^2$ occurs at $\alpha = 0$ since $||u - (v + \alpha z)||^2 \geq ||u - v||^2$ for all $\alpha \in \mathbb{K}$. Therefore, $||w - \alpha z|| \geq ||w||^2$ for all $\alpha \in \mathbb{K}$.

$$\begin{aligned} \text{Also, } ||w - \alpha z||^2 &= \langle w - \alpha z, w - \alpha z \rangle \\ &= \langle w, w \rangle - \langle \alpha z, w \rangle - \langle w, \alpha z \rangle + \langle \alpha z, \alpha z \rangle \\ &= ||w||^2 - \alpha \langle z, w \rangle - \bar{\alpha} \langle w, z \rangle + |\alpha|^2 \; ||z||^2. \end{aligned}$$

Consider $\alpha = \langle z, w \rangle \in \mathbb{K}$.

$$\begin{aligned} \text{Then, } \phi(\langle z, w \rangle) &= ||w - \langle z, w \rangle z||^2 \\ &= ||w||^2 - |\langle z, w \rangle|^2 - |\langle w, z \rangle|^2 + |\langle z, w \rangle|^2 \; ||z||^2 \\ &= ||w||^2 - |\langle w, z \rangle|^2 - |\langle w, z \rangle|^2 + |\langle z, w \rangle|^2 \; ||z||^2 \\ &= ||w||^2 - |\langle w, z \rangle|^2. \end{aligned}$$

However, since $\phi(\alpha) = ||w - \alpha z||^2$ has a minimum of $||w||^2$ at $\alpha = 0$, $\phi(\langle z, w \rangle) \geq ||w||^2$. Therefore, $|\langle w, z \rangle| = 0$, meaning $\langle w, z \rangle = 0$. Since $z \in M$ was arbitrary, $\langle w, z \rangle = 0$ for all $z \in M$.

To show uniqueness, suppose there exists v_1, v_2, w_1, w_2 such that $u = v_1 + w_1 = v_2 + w_2$ with $v_i \in M$ and $\langle w_i, z \rangle = 0$ for all $z \in M$ and $i = 1, 2$. Since $\langle w_i, z \rangle = 0$ for all $z \in M$ and $v_1, v_2 \in M$, we have that $0 = \langle w_1, v_1 \rangle = \langle w_2, v_1 \rangle = \langle w_1, v_2 \rangle = \langle w_2, v_2 \rangle$. Since $w_1 = u - v_1$ and $w_2 = u - v_2$, these equations give the following equations:

$$\begin{aligned} 0 &= \langle w_1, v_1 \rangle = \langle u - v_1, v_1 \rangle = \langle u, v_1 \rangle - \langle v_1, v_1 \rangle \\ 0 &= \langle w_2, v_1 \rangle = \langle u - v_2, v_1 \rangle = \langle u, v_1 \rangle - \langle v_2, v_1 \rangle \\ 0 &= \langle w_1, v_2 \rangle = \langle u - v_1, v_2 \rangle = \langle u, v_2 \rangle - \langle v_1, v_2 \rangle \\ 0 &= \langle w_2, v_2 \rangle = \langle u - v_2, v_2 \rangle = \langle u, v_2 \rangle - \langle v_2, v_2 \rangle. \end{aligned}$$

So, these equations give us that

$$\begin{aligned}\langle u, v_1\rangle &= \langle v_1, v_1\rangle \\ \langle u, v_1\rangle &= \langle v_2, v_1\rangle \\ \langle u, v_2\rangle &= \langle v_1, v_2\rangle \text{ and,} \\ \langle u, v_2\rangle &= \langle v_2, v_2\rangle.\end{aligned}$$

From the above four equations, the first two equations give that $\langle v_1, v_1\rangle = \langle v_2, v_1\rangle$, or equivalently, $\langle v_1 - v_2, v_1\rangle = 0$. The second two equations give that $\langle v_1, v_2\rangle = \langle v_2, v_2\rangle$, or equivalently, $\langle v_1 - v_2, v_2\rangle = 0$. So, $\langle v_1 - v_2, v_1 - v_2\rangle = 0$. Thus, from the positivity of an inner product, $v_1 - v_2 = \mathbb{O}$, i.e. $v_1 = v_2$.

Since $w_1 = u - v_1$ and $w_2 = u - v_2$, we can also rewrite the equations as follows:

$$\begin{aligned}0 &= \langle w_1, v_1\rangle = \langle w_1, u - w_1\rangle = \langle w_1, u\rangle - \langle w_1, w_1\rangle \\ 0 &= \langle w_2, v_1\rangle = \langle w_2, u - w_1\rangle = \langle w_2, u\rangle - \langle w_2, w_1\rangle \\ 0 &= \langle w_1, v_2\rangle = \langle w_1, u - w_2\rangle = \langle w_1, u\rangle - \langle w_1, w_2\rangle \\ 0 &= \langle w_2, v_2\rangle = \langle w_2, u - w_2\rangle = \langle w_2, u\rangle - \langle w_2, w_2\rangle.\end{aligned}$$

Once again, combining the first two equations (of the four above) yields that $\langle w_1, w_1\rangle - \langle w_1, w_2\rangle = \langle w_1, w_1 - w_2\rangle = 0$. Combining the second two equations yields that $\langle w_2, w_1\rangle - \langle w_2, w_2\rangle = \langle w_2, w_1 - w_2\rangle = 0$. So, $\langle w_1 - w_2, w_1 - w_2\rangle = 0$. Thus, from the positivity of an inner product, $w_1 - w_2 = \mathbb{O}$, i.e. $w_1 = w_2$. Therefore, $v_1 = v_2$ and $w_1 = w_2$, so the representation of u is unique.