Name: __________________________

Please check the appropriate box:

_____ I am taking this exam as a qualifier.

_____ I am not taking this exam as a qualifier.
Instructions:

1. You may use 1 formula sheet.

2. You must show your work in order to receive any credit. Answers without supporting calculations/justifications will be given no credit.

The following results may (or may not) be helpful in working the exam.

A) The probability mass function for a Poisson distribution with parameter $\theta$ (or $\text{Poi}(\theta)$) is

$$f(x; \theta) = \begin{cases} \frac{\theta^x e^{-\theta}}{x!}, & x = 0, 1, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

The maximum likelihood estimator of $\theta$ based on a random sample from this distribution is the sample mean.

B) The density function for the normal distribution with mean $\mu$ and variance $\sigma^2$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}, -\infty < x, \mu < \infty, \sigma > 0.$$ 

C) The density function for the gamma distribution is

$$f(x) = \begin{cases} \frac{1}{\Gamma(a)b^a} x^{a-1} \exp \left\{ -x/b \right\}, & x, a, b > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The mean and variance are $ab$ and $ab^2$ and the moment generating function is $E[e^{tX}] = (1 - bt)^{-a}$.

D) The probability mass function for a geometric distribution with parameter $\theta$ is

$$f(x; \theta) = \begin{cases} \theta(1 - \theta)^{x-1}, & x = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

The mean and variance are $1/\theta$ and $(1 - \theta)/\theta^2$. 
1. (20 points) Let \( X_1, \ldots, X_n \) be a random sample from a geometric distribution with parameter \( \theta \).
   a) (10 points) Find the complete and sufficient statistic for \( \theta \). Justify your answer.
   b) (10 points) Find the UMVUE of \( \theta^{-2} \). Justify your answer.

2. (35 points) Let \( X_1, \ldots, X_n \) be a random sample from the Poi(\( \theta \)) distribution. Consider estimation of the parametric function \( \tau(\theta) = e^{-2\theta} \).
   a) (5 points) Show that \((-1)^{X_1}\) is unbiased for \( \tau(\theta) \).
   b) (10 points) Derive the UMVUE of \( \tau(\theta) \) and justify your answer. [Hint: Use the Rao-Blackwell Theorem in conjunction with part a).]
   c) (10 points) Using a Gamma\((a,b)\) prior for \( \theta \), find the Bayes estimator (under squared error loss) for \( \theta \); i.e., find \( E[e^{-2\theta}|X = x]\).
   d) (5 points) What is the maximum likelihood estimator of \( \tau(\theta) \)? Justify your answer.
   e) (5 points) Give an argument that indicates why your answers to b), c) and d) should all be approximately the same for large values of \( n \).

3. (25 points) Let \( X_1, \ldots, X_n \) be a random sample from a Poi(\( \theta \)) distribution. Using a Gamma\((a,b)\) prior for \( \theta \), derive a 100\((1 - \alpha)\)% Bayesian credible interval for \( \theta \) based on percentiles from an appropriate chi-square distribution.

4. (20 points) Let \( X = (X_1, \ldots, X_n) \) be a random sample from a normal distribution with mean 0 and variance \( \theta \). Give the form of the UMP critical region of size \( \alpha \) (i.e., you must give an explicit form for the critical value that gives it the desired level) for testing \( H_0 : \theta = 1 \) versus the alternative \( H_1 : \theta < 1 \). Justify that the region is UMP.