

Theory of Statistics II: Final Exam

Name: _____

Please check the appropriate box:

I **am** taking this exam as a qualifier.

I **am not** taking this exam as a qualifier.

Theory of Statistics II: Exam 2

Instructions:

1. You may use 2 formula sheets.
2. You must show your work in order to receive any credit. Answers without supporting calculations/justifications will be given no credit.

The following results may (or may not) be helpful in working the exam.

- A) Let X_1, \dots, X_n be independent and identically distributed random variables with common pdf f and cdf F . Then, the density of the i th order statistic is

$$f_{X_{(i)}}(x) = \frac{n!f(x)}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i}.$$

- B) The density function for the gamma distribution is

$$f(x) = \begin{cases} \frac{1}{\Gamma(\nu)\theta^\nu} x^{\nu-1} \exp\{-x/\theta\}, & x, \nu, \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The mean and variance are $\nu\theta$ and $\nu\theta^2$. A chi-squared distribution with $2d$ degrees-of-freedom is a gamma distribution with parameters $\alpha = d$ and $\theta = 2$.

- C) The density function for the inverse gamma distribution is

$$f(x) = \begin{cases} \frac{b^a}{\Gamma(a)} x^{-(a+1)} \exp\{-b/x\}, & x, a, b > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Its k th moment is $E[X^k] = b^k / (a-1) \cdots (a-k)$ for $a > k$.

1. **(50 points)** Let X_1, \dots, X_n be a random sample from a gamma distribution with known shape parameter ν and unknown scale parameter θ .
 - (a) **(15 points)** Find the uniformly minimum variance unbiased estimator (UMVUE) of $\tau(\theta) = \theta^2$. State explicitly why you know your statistic is the UMVUE.
 - (b) **(10 points)** Compute the maximum likelihood estimator $\hat{\theta}$ of θ . What is the mle of $\tau(\theta)$ in this case. Justify your answer.
 - (c) **(15 points)** What is the limiting distribution $\sqrt{n} \left(\tau(\hat{\theta}) - \tau(\theta) \right)$?
 - (d) **(10 points)** Use the inverse gamma conjugate prior for θ to derive a Bayes estimator of θ under squared error loss; i.e., find $E[\theta_2 | X_1 = x_1, \dots, X_n = x_n]$. Then, give the form of a $100(1 - \alpha)\%$ credible interval for θ that can be used under this choice for the prior.

2. **(30 points)** Let X_1, \dots, X_n be independent and identically distributed as the random random variable X with probability density function

$$f(x|\theta) = \begin{cases} \frac{1}{\theta x^2} \exp\{-\frac{1}{\theta x}\} & x, \theta > 0, \\ 0 & \text{otherwise} \end{cases}$$

- (a) **(15 points)** Show that the uniformly most powerful α level critical region for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$ is

$$\left\{ x = (x_1, \dots, x_n) : \sum_{i=1}^n \frac{1}{x_i} \geq k \right\}$$

for some constant k . State explicitly why you know this region is uniformly most powerful against all alternative having $\theta > \theta_0$.

- (b) **(15 points)** Show that $1/X$ has an exponential distribution and use this fact to obtain an explicit expression for k in the case of an α level test using a percentile of a chi-squared distribution.

3. **(20 points)** Let X_1, \dots, X_n be independent and identically distributed as the random random variable X that has the probability density function

$$f(x|\theta) = \begin{cases} \frac{\theta}{x^2} & x > \theta > 0, \\ 0 & \text{otherwise} \end{cases}$$

Construct a $100(1 - \alpha)\%$ confidence interval for θ based on the statistic $Y = \min(X_i)$.