The generation and use of graphical examples in calculus classrooms: The case of the mean value theorem

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ABSTRACT

We analyzed video data of five instructors teaching the Mean Value Theorem (MVT) in a first-semester calculus course as part of a broader project investigating how active learning strategies were being implemented and supported in calculus courses. We sought to identify the ways examples of functions that did or did not satisfy the conclusion of MVT were generated and used in instruction. Using thematic analysis, we identified four themes that serve as characterizations of examples, which then allowed for the analysis of trends and patterns. We propose that attention to the generation and use of examples serves as one lens for considering how students can be engaged in the mathematical activity of the classroom, with implications for learning. This work contributes to an evolving notion of what is entailed in students’ active learning of mathematics and the role of the instructor in facilitating active learning opportunities.

1. Introduction

Although educational research has shown that students develop a deeper understanding of mathematics in classrooms where they are actively engaged, lecture is still the primary (and often only) mode of instruction in many collegiate-level mathematics courses (Freeman et al., 2014). In this paper, we highlight part of a broader project in which we studied five instructors of first-semester calculus to see how or if active learning was implemented in their classes. We analyze data from instruction covering the Mean Value Theorem (MVT), which says, "If $f$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval $(a, b)$, then there exists a number $c$ in $(a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$" (Larson & Edwards, 2015). A special case of the MVT where $f(a) = f(b)$ is stated as Rolle’s Theorem, resulting in the existence of a $c$ value where $f'(c) = 0$.

A key feature of the set of MVT lessons was the emergence of many graphical examples of functions provided by instructors or students that satisfied or did not satisfy the conclusion of the MVT. As part of a more general interest in considering active learning in calculus, its form, and how particular features can support student engagement and learning, we were drawn to this common feature of examples. As such, we sought to answer the research questions: In what ways are examples generated and used in instruction focused on defining and illustrating the MVT? What role do these examples play in contributing to an active learning environment? Grounded in a social theory of learning (Wenger, 1998), we present thematic analyses of video recorded lessons from which we identified four themes that serve as ways to characterize different aspects of the emergence and use of examples in mathematics classrooms. We first define the four themes then illustrate their interaction using vignettes from a subset of the lessons in our data.

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Through this, we propose that attention to the generation and use of examples serves as one lens for considering the nature of the opportunities for students to be engaged in the mathematical activity of the classroom, thus contributing to their learning. We offer a set of implications for teaching and additional considerations for how active learning and attention to the generation and use of examples can inform not only student participation but also the depth of and connections across mathematical ideas in classroom discussion.

2. Literature review

The phrase “active learning” has become a predominant feature of conversations around the reform of undergraduate mathematics (and, more broadly, STEM) education (e.g., Conference Board of the Mathematical Sciences, 2016) and has been attributed to improved educational environments (e.g., Freeman et al., 2014). However, a sense of what is meant by active learning in mathematics, and how to study its impact, is wide-ranging in terms of focus and specificity. In this section, we highlight the ways in which active learning is discussed in the literature. From this synthesis, we put forth how we intend to use the phrase, connect it to our theoretical perspective, and motivate our focus on the generation and use of examples in mathematics classrooms.

2.1. Active learning in mathematics

In a position statement from Conference Board of the Mathematical Sciences (2016), active learning in mathematics is defined as “classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving, that promote higher-order thinking” (p. 1). This is consistent with other definitions of the phrase (e.g., Freeman et al., 2014; Tsai et al., 2013) that focus on the structure of classroom activity and the nature of student participation. There is a not a pedagogically unique definition of active learning and, instead, a range of strategies that apply, including approaches such as “inquiry-based learning” (Kogan & Laursen, 2014; Laursen, Hassi, & Hough, 2016), “flipped classrooms” (Code, Piccolo, Kohler, & MacLean, 2014; McCallum, Schultz, Sellke, & Spartz, 2015), multimedia projects (Hulsizer, 2016), and modeling activities (Tsai et al., 2013). As such, efforts toward promoting active learning often focus on incorporating classroom strategies and structures that are deemed to involve more active participation from students, which greatly reframes how class time is spent in mathematics classrooms and the role of the instructor in facilitating classroom instruction. These efforts require great investment, as has been highlighted by the research team on the NSF-funded “Student Engagement in Mathematics through an Institutional Network for Active Learning” (SEMINAL) project (Association of Public & Land-Grant Universities, 2018), including funding, dedicated and specialized classroom space, extended class time, faculty collaboration, course coordination, and instructor development.

The investment is seen as worthwhile based on the documented effects of active learning approaches. In general, active learning environments have been found to have a positive impact on undergraduate students’ achievement in mathematics and their dispositions toward mathematics, which has implications for retention in and completion of STEM degrees (e.g., Code et al., 2014; Freeman et al., 2014; Rasmussen & Kwon, 2007). This has been found to be particularly true for students from underrepresented populations (Laursen, Hassi, Kogan, & Weston, 2014; Treisman, 1992) and students labeled as “low achievers” (Kogan & Laursen, 2014). Additionally, these gains have been found to be long-term throughout subsequent mathematics courses (Kogan & Laursen, 2014). Students have also been found to recognize and value the novel aspects of active learning approaches (e.g., Larsen, Glover, & Melhuish, 2015; McCallum et al., 2015).

However, issues remain regarding the impact and implementation of active learning approaches in undergraduate mathematics contexts. Recent studies have highlighted continued questions about the benefits and substance of approaches deemed to be more engaging for students (e.g., Sonnert & Sadler, 2015). For example, in her study on the effects of an assignment for students to create videos as part of exam preparation in differential equations and complex analysis courses, Hulsizer (2016) found that students expressed a positive attitude, in general, about the assignment and classified it as different from traditional class work. However, students indicated a preference for instructor-led review. Recent critiques have also examined the disconnect between active learning approaches and more equitable student outcomes (Johnson et al., 2018; Kuster, Johnson, Keene, & Andrews-Larson, 2018). These examples raise questions about the way in which structural changes to the mathematics classroom result in improved learning outcomes or in shifting conceptions about what it means to learn and do mathematics and who is able to participate in that activity.

Furthermore, though many mathematics departments have identified the inclusion of active learning strategies as key to successful undergraduate mathematics courses, it is not a consensus view (Bressoud & Rasmussen, 2015). Even among departments that value incorporating active learning approaches, most report not being successful at doing so. As a result, lecture formats and other approaches that would be deemed to be more passive in terms of student engagement are still the predominant approach in many college mathematics courses (Freeman et al., 2014).

With these continued concerns, it is important to recognize that active learning in mathematics entails more than structural changes and new instructional strategies. Some scholars have defined active learning in ways that highlight these other changes. For example, Larsen et al. (2015) highlight a shift in the goals of teaching calculus, building on the work of Lampert, Beasley, Ghousseini, Kazemi, and Franke (2010) in K-12 teaching, toward a more robust set of mathematics proficiencies—conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions (Kilpatrick, Swafford, & Findell, 2001). This shift highlights active learning in mathematics as more than just a set of instructional features and, instead, emphasizes a reframing of the goals for mathematical learning that can guide teaching decisions. Other scholars have highlighted the need for active learning approaches to instill in students the thought that mathematics is a creative human endeavor to which they can contribute (Rasmussen & Kwon, 2007) and that these approaches should be responsive to student thinking and should position...
student reasoning and contributions as central to the mathematical work of the classroom (Association of Public & Land-Grant Universities, 2018).

We find these perspectives to be useful in driving efforts toward reform as it moves the considerations beyond a focus on classroom features and toward articulating goals and overarching considerations for the approach. We also find it useful to consider what is meant by “engagement” as part of the goal of active learning approaches. While the idea of engagement in mathematics can be taken to mean a variety of things, we follow the conception of engagement put forth by Middleton, Jansen, and Goldin (2017) as, “the in-the-moment relationship between someone and her immediate environment, including the tasks, internal states, and others with whom she interacts” (p. 667) that is inseparable from learning. Furthermore, Middleton et al. (2017) highlight the complexity of students’ engagement—a dynamic system of motivation, emotion, and social interaction.

A theoretical perspective on student development that is consistent with that vision is needed to further support efforts to incorporate and learn about active learning in undergraduate mathematics—to inform both research and design. From our review, much of the work highlighting active learning in mathematics lacks a clear articulation of a theory of learning. When theoretical perspectives are more explicitly outlined, they tend to focus on cognitive perspectives around engagement or the role of interaction and discovery on an individual’s construction of knowledge. These perspectives are important for understanding student development and the role of classroom environments on that development. However, with a focus on a broader conception of mathematical proficiency and the social nature of engagement—students’ own sense of their agency and authority in constructing mathematical knowledge and the way in which students’ ideas are positioned in the classroom context—we assert that a theoretical perspective that considers not only the role of social contexts and interactions on individual learning but also the social nature of learning itself is beneficial to further examine the nature of active learning efforts.

2.2. Theoretical framework: Wenger’s social theory of learning

In order to attend to the aspects of social interaction that are part of a complex system of student engagement (Middleton et al., 2017), we frame our work considering active learning in the undergraduate mathematics classroom using Wenger’s (1998) social theory of learning, often linked to a “communities of practice” perspective. From this perspective, the mathematics classroom can be seen as a community that shapes individual students’ sense of what it means to learn and do mathematics and also shapes students’ opportunities to engage mathematically. This shaping can occur in productive or less-than-productive ways. For example, lecturing as a primary approach in a mathematics classroom structures the relationship between students, tasks, and the classroom environment in ways that position students only as recipients of established mathematical ideas. On the other hand, when students have opportunities to contribute ideas, explain their reasoning, and interact with and consider the contributions and reasoning of others, students have the opportunity to productively engage both cognitively and socially (Fredricks, Blumenfeld, & Paris, 2004; Middleton et al., 2017).

While pedagogical features attributed to active learning, such as small group work, are intended to provide these opportunities, we contend that there also should be a focus on mathematical practices, such as seeing structure in mathematics, the collaborative pursuit of mathematical discovery, and proof and justification. In this paper, we consider these to be “authentic” mathematical practices, and we use that word to characterize activity consistent with those practices. This notion is juxtaposed with less authentic practices, such as copying notes and solving simple or prescriptive problems that are either inconsistent with how people engage in mathematical work or are limited in their scope. While courses like those in the calculus sequence are not only designed to prepare students for academic mathematics careers, engaging students in authentic mathematical practices is key to developing robust procedural and conceptual understandings.

Wenger (1998) integrated four components of learning—meaning, practice, community, and identity—that we consider as intertwined with the idea of engagement. A productive mathematics classroom is one in which students have the opportunity to learn mathematics. This means that students have opportunities to: experience ways of doing and constructing mathematics (meaning), to engage in mathematical practices (practice), to be positioned in the classroom community as competent participants in mathematical activity (community), and to come to see themselves (and be seen by others) as ones who do mathematics (identity). This highlights issues with conceptions of active learning that only focus on “participation”—such as opportunities for working in small groups or monitoring air time in whole group contexts. The substance of that participation and how students are ultimately positioned in the midst of that participation is equally important. We contend this perspective serves as a useful lens for understanding the structural elements and characteristics of supporting active learning in the undergraduate mathematics classroom.

2.3. The role of examples in learning mathematics

One way in which students can engage with and construct mathematical ideas in the classroom context is through the generation and use of mathematical examples. Watson and Mason (2005) provided a broad definition of example encompassing six categories: (1) illustrations of concepts and principles, (2) questions worked out in a textbook or by a teacher, (3) questions to be worked on by students as a means of learning to use, apply, or gain fluency with specific techniques, (4) representatives of classes used as raw material for inductive mathematical reasoning, (5) contextual situations that motivate concepts, and (6) prototypical cases that illustrate general definitions or theorems (Watson & Mason, 2005, p. 3). Students who have opportunities to reason about, offer, and make connections among mathematical examples, and to develop a clear sense of the way in which examples serve a collective effort to build mathematical ideas are positioned to be productively engaged in mathematical activity. This is, in part, based on the way in which the generation and use of examples have been connected to mathematical practices (e.g., Iannone, Inglis, Mejía-Ramos,
While many students view examples provided by teachers and texts as templates for solving homework exercises (2004, Lithner, 2003), examples can play an important role in developing understanding of concepts. Watson and Mason (2005) introduced the notion of learner-generated examples and advocated for their power as a tool for deeper learning. Mason and Watson (2008) elaborated:

... when a teacher offers an example and works it through, it is the teacher’s example. Learners mostly assent to what is asserted. ... When learners construct their own examples, they take a completely different stance towards the concept. They ‘assert’; they actively seek to make sense of underlying relationships, properties, and structure which form the substance of the theorem or concept. (p. 200)

Mason and Watson (2008) noted “Learners who are encouraged to be creative and to exercise choice respond by becoming more committed to understanding rather than merely automat ing behavioural practices” (p. 192). To promote creativity in learner-generated examples, students should be encouraged to consider variation. That is, students need to be comfortable asking and exploring questions such as: “What can vary in this problem?” and “To what extent can this aspect of the problem vary?” Watson and Shipman (2008) noted:

...if students generate examples, reflection on those examples could, through perceiving the effects of the variations they have made, lead to awareness of underlying mathematical structure. ‘Structure’ here means how elements and properties of mathematical expressions are related to each other. (p. 98)

They further indicated that directed example generation, rather than “directionless exploration,” can be a good way to begin understanding concepts.

Through learner-generated examples, a personal example space is constructed and developed. A personal example space is defined to be the set of available examples and methods of example construction a learner has at their disposal for solving problems. Sinclair, Watson, Zazkis, and Mason (2011) examined how personal example spaces are structured, paying attention to the varying degrees of “connectedness” such personal example spaces may have. The more connected one’s example space, the greater the likelihood of having a stronger understanding of the concept. They indicate that slightly different prompts may trigger the use of different examples. Others (e.g., Fukawa-Connelly & Newton, 2014) have considered the extent to which examples that emerge in classroom discussion play a role in the construction of individual example spaces, though with less attention to students’ engagement in the mathematical activity of the classroom.

We draw on Watson and Mason’s (2005) definition of example described previously to examine the use of examples in five instructors’ lessons on the MVT. In this paper, we focus on the particular aspects of Watson and Mason’s definition on illustrating and representing mathematical ideas, specifically the MVT. We argue that how examples are generated and used provide or restrict opportunities for students to be actively engaged in mathematical activity. Investigating the generation and use of examples can serve as one way to further understand how active learning is supported in the mathematics classroom. In this study, we sought to answer the questions: In what ways are examples generated and used in instruction focused on defining and illustrating the MVT? What role do these examples play in contributing to an active learning environment? In the sections that follow we describe how the data was collected and analyzed, discuss the themes that emerged, and share detailed vignettes of two distinctly different lessons that highlight how the use of examples embodied the classroom culture.

3. Methods

3.1. Context and participants

Subjects in this study were five instructors of first-semester calculus at a large public research university. Author 1 served as the coordinator of the course as well as one of the instructors in the data set. During the semester in which data collection began, all instructors who were teaching first-semester calculus were invited to participate in the study, and four instructors (plus the author) chose to participate. Participants were told that the purpose of the study was to examine how active learning was implemented in various instructors’ classes. No requirements were set as to how much or what type of active learning instructors would need to implement, but instructors knew that the overall project had a goal of increasing the amount of active learning in entry-level mathematics courses. To preserve confidentiality, we use the term instructor to describe the instructor of record of the course, regardless of whether the instructor was a tenured faculty member, a full-time teaching instructor, or a graduate student. Furthermore, we use the pronouns she, her, and hers to describe all five instructors and refer to them in this paper as Instructor A, Instructor B, etc. All subjects consented to the study, and all but the author (Instructor A) received a $500 stipend for their participation at the completion of each semester of the project. Additionally, students in each class were asked to sign a media release form granting permission to use their image or voice in our data.

The course in which data was collected was a coordinated course, with four common exams and a common final exam. A sample calendar was provided to all instructors by the coordinator, but instructors were free to rearrange the calendar, as needed, provided that they cover the required material prior to each exam. All instructors of the course met weekly to discuss upcoming topics and, on occasion, decided as a group to move topics from one exam to the next. A few of the topics in the course were highly coordinated, in that the coordinator chose the course materials that students would work on during the class period. These activities included a one-day lesson introducing limits, a two-day lesson introducing derivatives, and a three- to four-day lesson introducing Riemann sums and
definite integrals. For all other class sessions, each instructor chose how he or she would cover the material. The coordinator, Instructor A, shared her course materials in a collective cloud folder. All other instructors were free to use that material but were not required to do so.

3.2. Data collection

During the first semester of the project, we video recorded class sessions of the five participants, starting in the third week of classes. All sessions that covered new material were recorded excluding sessions during which students were reviewing for an exam or taking a quiz or an exam. In each classroom, a video camera was placed in the back corner and was focused on the instructor during the class period. Most of the time, the video captured a wide angle, so many of the students in the classroom appear on camera. During the second semester, two of the five instructors were recorded every day, and the other three instructors were video-recorded when teaching two units—one on the MVT, which was not coordinated, and one on definite integrals, which was highly coordinated. In this paper, we discuss data from the MVT sessions during the second semester. We purposefully selected data from the un-coordinated sessions because this provides an example of instruction in these classrooms without the influence of the coordinated lessons. Three of the instructors covered the material in one day of class, and two of the instructors used two partial days of class. Amount of class time spent introducing the MVT is shown in Table 1.

<table>
<thead>
<tr>
<th>Instructor</th>
<th>Class Time (in minutes)</th>
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<tbody>
<tr>
<td>A</td>
<td>42</td>
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<tr>
<td>B</td>
<td>53</td>
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<tr>
<td>C</td>
<td>40</td>
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<tr>
<td>D</td>
<td>48</td>
</tr>
<tr>
<td>E</td>
<td>38</td>
</tr>
</tbody>
</table>

3.3. Data analysis

As part of a larger project, we used thematic analysis, a "method for identifying, analysing, and reporting patterns (themes) within data" (Braun & Clarke, 2006, p. 79). We employed theoretical thematic analysis, which is "driven by the researcher's theoretical or analytic interest in the area, and is thus more explicitly analyst-driven" (Braun & Clarke, 2006, p. 84). Initially, our focus of the analysis was on ways in which active learning was being used in the classroom, focusing on structural features of instruction. As such, we were using theoretical thematic analysis to code for times when students were working in groups or were actively participating in doing mathematics. Moreover, the social theory of learning perspective requires that we look not only at the ways in which students are participating, but in the ways that they were engaging in mathematics. Thus, we employed theoretical thematic analysis to identify these instances.

To better understand how students were engaged, we created timelines of instructional episodes to indicate when ideas were brought into the discussion, by whom (students or instructors), and how those ideas were taken up in the collective discussion. This allowed us to examine sequences of ideas to better illustrate who was participating and in what ways. From this analysis, we found instructors’ use of examples to illustrate or represent the MVT, as well as their take up of students’ examples, to be of particular significance. These included examples meant to illustrate properties related to the MVT, including counterexamples. In our data, the majority of these examples were graphical examples, and there was significant variation in how these graphical examples were used. As such, we began to focus our analysis on these graphical examples.

Videos of each instructor’s MVT lessons were viewed again independently by at least two researchers. We took an additional precaution by ensuring that Author 1 did not analyze her own video (Instructor A). In this viewing, each researcher created a record of all examples occurring in the lesson, noting when an example occurred (what point in the lesson), who initiated the example, who discussed the example, and the content of those discussions. After watching a video, individual researchers wrote analytic memos describing emerging ideas related to the generation and use of examples in the lessons. These served as points of discussion for the entire research team as we continued developing and refining themes emerging from the data, looking both within and across individual instructor’s lessons. If needed, we returned to the videos to re-watch the lesson to thoroughly capture ideas resulting in our discussions. This iterative process of individual and collective discussion and memo writing provided transparency to address potential biases regarding the analysis of Instructor A’s video.

Multiple passes through these instances yielded four themes regarding the generation and use of examples. These themes serve as ways to characterize aspects of the emergence and use of examples in the analyzed instances and manifested as a set of questions: (1) who generated the example?, (2) how precise was the example?, (3) who responded to the example?, and (4) how was the example used? Each identified example across the five lessons was characterized in these four ways. We then looked for trends within and across themes and examples to determine common ways in which examples were generated and used in each lesson.
4. Findings

In this section, we first provide a general overview of the five lessons. We then summarize the four themes that emerged from our data, each of which characterize features of how examples were generated and used in the classroom discussion. To further illustrate the interactions within the four themes, we then provide detailed descriptions of two of the classrooms as vignettes. These highlight the ways in which the collection of examples that emerged and were used in the context of classroom discussion opportunities to engage students in mathematical activity.

During instruction on the MVT, only Instructor A required students to work in groups during the development of the MVT. Instructors B, C, and D asked their students to work in groups to complete worksheets related to the MVT following the lecture on the topic. The worksheets provided practice problems similar to what had been done by the instructor and did not introduce any new material. Instructor E did not have her students work in groups at any point during the lesson. During all of the lessons for Instructors B, C, D, and E, there were many times where the instructor asked students to participate in some way, usually by answering a simple question or verifying that they understood something that was said. Instructors B, C, and E chose to introduce Rolle’s Theorem prior to the MVT, while Instructors A and D presented the MVT first, with Rolle’s Theorem given as a special case of the MVT. In every lesson, at least five graphical examples were utilized.

4.1. Who generated the example?

The first theme evident in our data relates to who generated an example. In all of the classrooms, there were times during a class session when the instructor provided an example for the class and wrote it on the board. We refer to these instances as instructor-generated examples. A second classification is what we referred to as instructor-interpreted examples. In these instances, the instructor asked students to provide an example of a function that met specific criteria, to which a student might respond with a short one- or two-word response that offered a specific characteristic of a possible example, but not a specific example. The instructor would then expand on the response and sketch a graph on the board. For example, during Instructor B’s lecture, the instructor asked the students if it was possible to draw a graph that was continuous but did not have a horizontal tangent. A student responded with “points,” and the instructor drew a graph on the board that resembled \( f(x) = |x| \). In this case, the instructor interpreted the student response of "points" to mean a function that had a cusp or corner, and the instructor took on the work of making sense of that idea, formulating an example, and sketching it on the board. In these cases, it often seemed that the instructors were expecting a specific response to their questions and, once a reasonably close response was given, instruction proceeded.

Finally, we classified examples that were created by students as learner-generated examples to be consistent with the work of Watson and Mason (2005). These examples usually were given in response to questions that had a broader response space where there existed several possible correct answers, and students were given the opportunity to formulate ideas and share reasoning. In our data, we saw several learner-generated examples in Instructor A’s class, one in Instructor B’s class, but none in the other classes. In Instructor A’s class, these examples were generated when students were given a prompt by the instructor to create several examples of graphs that did or did not satisfy a list of properties, followed by time to work in groups or to work independently at their seats. The students then placed these examples on the board. More details about these examples will be provided in the classroom vignette described in Section 5.1. The only other case of a learner-generated example was during a lecture when Instructor B asked for an example of a graph that met certain criteria:

Instructor B: Can I make a function that doesn’t have any horizontal tangent lines? Don’t necessarily worry about the domain for the moment.

Student 1: Can we think of one with horizontal tangent lines?

Instructor B: Um, without horizontal tangent lines. So, that’s going to be my goal right now, without horizontal tangent lines [points to where that’s written on the board]

Student 2: \( y = x \)

Instructor B: So, \( y = x \) works right [draws graph on the board]…That is a function with no horizontal tangent lines.

While this is still a brief contribution from a student, we claim that the instructor did not need to interpret the meaning of this example and, instead, was able to sketch the student’s desired graph on the board.

4.2. How precise was the example?

A key feature of each example that was generated and shared during classroom discussion (either by the instructor or by students) was the precision with which it was formulated and communicated, beyond whether the contribution was correct or incorrect. In our data, we identified no examples that we considered to be completely incorrect, but there were some that were imprecise or incomplete. For example, when Instructor A’s students were sketching their learner-generated examples on the board, some of the students provided incomplete examples by not labeling the endpoints of the interval with a and b. In this case, the instructor added these labels while addressing the importance of labeling key information on their sketches. In another instance, Instructor D provided a graphical example (Fig. 1)²

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¹ We use numbers to denote multiple students speaking in a given transcript excerpt (e.g., Student 1, Student 2, etc.). We repeat numbers in different transcripts, meaning that “Student 1” in different transcripts may not be the same individual.
These instances are not necessarily errors in reasoning but in the communication of ideas. It could be perceived as a missed opportunity to provide additional reasoning or clarify an idea, or merely an oversight. However, these contributions may indicate that one’s conception does not account for certain cases or conditions or may lead others to do so. For instance, in the example from Instructor D’s class described above, the imprecision in the sketch was not commented on in the class, leaving students with a graph that, taken at face value, does not meet the conditions to be met in a discussion of the MVT. Had a student drawn such a graph on an exam as an example of a function that satisfies the MVT, he or she would not receive full credit. This is striking, as at some point during instruction each instructor took the time to address other common exam pitfalls.

4.3. Who responded to the example?

While our first two themes focused on the emergence of examples in classroom discussion, the next two themes are focused on what happens after that initial contribution. In this section, we examine whether the instructor or the students were engaged in responding to an example after it was initially presented. Across our data, examples were primarily commented on by the instructor or not verbally addressed at all. In some cases, the response from the instructor took the form of posing additional questions to clarify aspects of the contribution that were not clear. For example, when students were sketching their examples on the board in Instructor A’s class, one student drew an example (Fig. 2, left) that was not clear. The instructor asked the students who generated the example if they meant for there to be a vertical asymptote on the graph, and after receiving verbal clarification, Instructor A modified the graph to the one in Fig. 2 (right). More details about these scenarios are in the vignette for Instructor A in Section 5.1.

Another way that examples were responded to by instructors was the explicit or implicit determination of whether to take up or disregard a contribution. For example, when Instructor D asked her students for an example with specific constraints, a student responded with, “asymptotes.” This is how the episode transpired in class:

Instructor D: What would cause it to fail? Why wouldn’t it work?
Student 1: [incomprehensible]
Instructor D: Ok, asymptotes. What condition are you attacking with that?
Student 1: [inaudible]

\(^2\) All figures have been recreated from examples presented in class.
Instructor D: Discontinuity. Or, you could attack which other one?

Student 2: Differentiability.

Instructor D: Differentiability. Right? The derivative. So, let’s take a look here. Let’s look at a graph much like this one. Actually, I’ll draw the exact same one [draws Fig. 1 again]. There’s my average or my secant line. And it’s supposed to happen up here and down here [points to max and min], right? What would happen if I put a hole there?

Student: [inaudible]

Instructor D: It’s not differentiable. Is it not continuous either? Yeah. It’s not continuous.

We observe that Instructor D noted that the student was addressing continuity and drew an example that was not continuous, but did not have an asymptote, as suggested by the student. The graph she drew was the same as her original graph [Fig. 1], but with the addition of holes at the maximum and minimum. However, she never followed up with an example that took up the first student’s idea of asymptotes.

In our data, we had few instances during which students explicitly responded to specific generated examples. This primarily happened in Instructor A’s class, in which students generated examples in small groups. After the students put their examples on the board, they were given an opportunity to respond to each other’s examples and discuss whether or not they were valid. However, even in this case, the instructor settled a disagreement and explained why a given graph was a valid example. More details about these scenarios are in the vignette for Instructor A in Section 5.1.

We observed only one instance of a student responding to an instructor-generated example. This occurred in during the lesson from Instructor B, who was one of the instructors who introduced Rolle’s Theorem prior to introducing MVT. In an example used to begin to illustrate the MVT, for which \( f(a) \) does not need to equal \( f(b) \), the graph drawn happened to have a point at which the tangent line was horizontal, though that was not the point of interest (Fig. 3). A student asked, “It can’t be the same line can it? Right?” This prompted the instructor to draw the example in Fig. 4. Since this graph did not have a horizontal tangent anywhere on the graph, a student questioned whether or not all examples needed to have a horizontal tangent, and the instructor emphasized that

Fig. 3. Graph sketched by Instructor B to illustrate MVT.

Fig. 4. Linear function sketched by Instructor B to illustrate MVT.
the key idea was that the slope of the secant line needed to be the same as the slope of the tangent line. This exchange is presented in the excerpt below:

Student: But there’s no horizontal asymptote.
Instructor B: Um, horizontal tangent line, you mean?
Student: Yeah.

Instructor B: Yeah, so this is, so we are no longer guaranteed to have a horizontal tangent line. Clearly that doesn’t happen here [pointing to Fig. 4 on the board], right? What we’re guaranteed of is having, being able to draw a tangent line that’s parallel to our secant line. So, it has the same slope. Does that make sense? So, in this case we coincidentally ended up with a horizontal tangent line, but that doesn’t have to happen but what does have to happen is that we have to be able to draw a tangent line that’s parallel to our secant line.

While we hope that students were internally making sense of the examples generated across these lessons, the instance above was the only time we had evidence of this happening verbally and publicly. While it was a minor contribution, it highlighted a way in which students can be involved in the work of responding to examples and in the construction of mathematical ideas, even when the instructor is the primary source of new examples. It also highlights that the evaluation and critique of contributions does not need to be reserved for examples that are incorrect or in need of modification.

However, for many of the examples generated or interpreted by the instructor, students did not offer any responses, critiques, or evaluations. Again, we recognize that the lack of response from students in the moment is not necessarily a sign that students agreed with all of the examples across the data or that they were not making sense of the examples internally. It was also not entirely for lack of opportunity, as there were times when the instructors asked the students, "Does this work?" or "Does this make sense?", providing opportunities for students to ask questions or raise concerns, but we saw no evidence of students responding to those prompts in our data. Despite these varying explanations and factors, these trends raise questions about the way in which students are engaged in the mathematical work of the classroom.

4.4. How was the example used?

A final theme that emerged from our analyses was the timing and methods through which an example was elicited and drawn upon as part of the construction of mathematical ideas throughout a lesson. In this section, we describe some of the ways graphical examples were used in our data. The most common use of examples was to demonstrate an idea or a property. These tended to be along the lines of "existence proofs" where one example was enough to demonstrate that something was possible. Instructor E’s examples were all classified as these types of "existence proofs" as well as most of Instructor B’s examples. Often these were in response to instructor-posed questions such as, "Can we draw a graph with these constraints?" In these cases, one example was sufficient to prove that such tasks were possible. These examples were generated or interpreted by the instructor and sometimes part of what seemed to be a predetermined set of ideas and exemplars to present to students.

In Instructor B’s class, these "existence proof" examples were used in a sequence to lead students to Rolle’s Theorem, and eventually the MVT. She demonstrated this with a sequence of graphs with varying restrictions, starting by asking if it was possible to graph a function without a horizontal tangent line. Once a student had responded with \( y = x \), she included the restriction that \( f(a) = f(b) = 3 \). She continued with a sequence of more restrictive conditions, leading to the statement of Rolle’s Theorem, and eventually the MVT. Instructor E’s lesson used a similar progression and is illustrated in a classroom vignette in Section 5.2.

Instructors A and C used a collection of examples to illustrate the MVT. In Instructor A’s class, these were learner-generated examples that did not satisfy the conclusion of the MVT and were used to infer the conditions of the MVT as detailed in Section 5.1. In Instructor C’s class, the examples were instructor-generated examples that met the conditions of Rolle’s Theorem and were used to illustrate that all of the graphs had a horizontal tangent line.

At times a specific example was used to address a common misconception. For instance, from our experience teaching calculus, we know that students often mistakenly think that a linear function does not satisfy the conclusion of the MVT. Most of the instructors in our study explicitly addressed this common misconception. In Instructor A’s classroom, one group of students generated a linear function as a non-example, and Instructor A put it on the board (Fig. 5), instructing the class to make sure they discussed a graph like that in their groups. Later, she facilitated a discussion of this example with the class. Instructor B asked if it was possible to draw a graph that would not have a horizontal tangent line and satisfy the conditions \( f(-2) = f(2) = 3 \), \( f \) is continuous, and \( f \) is differentiable. After considering cases in which the graph started up and to the right from the left endpoint and then down and to the right, she highlighted the third case of a graph that goes “straight across.” She emphasized that the last case has horizontal tangent lines at every point on the interval. Instructor E used similar reasoning with her students, but never provided a graphical example, instead using hand gestures to convey the ideas. These cases, as well as the example from Instructor B’s class with a positive slope (Fig. 4), highlight the way in which examples can address common misconceptions that might otherwise persist.

Beyond the use of examples to address various cases of a given concept, some instructors used examples as a discussion point for common exam errors. Instructor A noted that a graph drawn by a student (Fig. 6) would not receive credit on an exam as an example that did not have a horizontal tangent because it inadvertently shows two points at which the tangent line is horizontal. This particular example was one in which the students seemed to understand the concept, but the precision of the sketch was not up to par. The instructor modified the graph (Fig. 6) to show how to make the students’ intent obvious, highlighting the need to be careful when drawing their graphs. The generation of examples that avoided these kinds of pitfalls was common during Instructor E’s lesson, who directed her students to keep their graphs simple. She provided two graphs—one as an example that was overly complex and contained a flaw that would lead to a deduction in points on an exam, and one that was more straightforward in satisfying the
demands of a potential exam question (Fig. 7). The idea seemed to be that the simpler the graph, the less likely a student would be to make a mistake.

5. Illustrative vignettes

In this section, we provide two vignettes describing the MVT lessons of Instructors A and E. We chose these two vignettes as they represent an interesting set of distinctions that arose in the data. Instructor A’s lesson featured the majority of learner-generated examples. Instructor E’s lesson contains a general lack of learner-generated examples and in this way is similar to the lessons of Instructors B, C, and D. However, Instructor E’s lesson featured fewer mathematical imprecisions than other instructors’ lessons and a novel set of purposes for particular examples, such as using “simple” examples for the sake of performing well on exams. We feel these two vignettes serve as interesting cases to present in more detail and to compare to one another. In each vignette that follows, we illustrate how the four themes used to characterize the emergence and use of each example interact and play a role in fostering or constraining the nature of the opportunities students have to engage in the development of the Mean Value Theorem.

5.1. Classroom vignette: Instructor A

At the beginning of Instructor A’s lesson on the MVT, a graph (Fig. 8) was provided to students as an instructor-generated example. The instructor then asked her students to tell her what a secant line was (several students responded), and she drew the secant line on the graph between the two endpoints. Next, she told her students to work at their seats to see if there was any place on
the graph where there was a tangent line with the same slope as the secant line and, if so, to sketch the tangent line at that point on their own paper. For just over two minutes, the instructor walked around the room, looking at the work done by the students and clarifying directions for students who had questions. It seemed to be expected that every student would participate, and from what was visible in the video, every student did so.

After a brief class discussion about the previous graph, during which relevant points were identified, Instructor A told the class to work in groups to see if they could find examples of other graphs where there was or was not a tangent line with the same slope as the secant line between the two endpoints. She instructed her students by saying, "Your next job is to make sure you find some graphs that do have this property and some graphs that don't have the property." Students spent approximately 16 minutes working in groups to create several learner-generated examples that satisfied the property and several that did not satisfy the property. At one point, the instructor put one of the student's examples on the board (a linear function; Fig. 5 above) and told the class to make sure they discussed an example like this one, if they had not already done so. After a short time, while students were still working in groups, she asked the class if they thought the graph of a linear function had the property or not and got different answers from two groups. The class discussion proceeded as follows:

Instructor A: So, does this one have the property or not (pointing to graph of horizontal straight line on board)?
Student 1: Not.
Student 2: Why not?
Students (multiple): Yeah. [many talking over each other]
Student 4: Everywhere.
Instructor A: Everywhere. So, my secant line is this exact same thing. Right, I can draw a line between these two points. I've already done it. What's the tangent line look like here?
Students (multiple): same thing
Instructor A: Same thing. And here? Same thing (students echoing same thing)
The students in the group that was incorrect quickly changed their minds and then correctly explained that every point on the interval would be a place where the tangent line had the same slope as the secant line.

After it was clear that every group had several learner-generated examples that had the property and several that did not have the property, Instructor A asked each group to send at least one person to the board to sketch an example of a graph that did not have the property (i.e. a graph where there was no tangent line with the same slope as the secant line between the endpoints). Nine graphs were drawn on the board by the students, and Instructor A added one more graph (Fig. 9c) that was used by one of the groups, but was not the one they chose to put on the board. Thus, there were ten learner-generated examples on the board (shown in Fig. 9, with graphs labeled Fig. 9a through j).

Instructor A made a few small modifications to the learner-generated examples, mainly by labeling $a$ and $b$ if they were omitted by the students. Additionally, she drew the vertical asymptote in Fig. 9h, as discussed in Section 4.4. In all of these cases, we see evidence that Instructor A evaluated the learner-generated graphs, specifically attending to precision.

Next, the class was instructed to look at all of the graphs to see if there were any graphs that should not have been on the board, in other words, to see if any of the graphs on the board had a place where the slope of the tangent line was equal to the slope of the secant line. This created an opportunity for the students to evaluate the validity of the examples. There are a few noteworthy instances of how examples were evaluated during this discussion. First, immediately after the instructor posed the question, a student said that Fig. 9a was wrong. The speed at which he answered makes us believe that he had already been evaluating the examples prior to the instructor asking the class to do so:

Instructor A: Take a minute and look at those. See if you agree or disagree that those do not satisfy the property that we want them
Instructor A: Top right. What do you think about the one on the top right? This blue one here?

Student 1: Yeah.

Instructor A: So where has it satisfied the property?

Students (several speaking at once): Zero

Instructor A: So we have satisfied the property over here, but we really only care about the interval from \(a\) to \(b\). So, if we really only have the stuff from \(a\) to \(b\) [erasing everything outside the interval] and forget the rest of the graph, how about now?

Students (several): Yeah.

This student argued that the graph shown in Fig. 9a was wrong because there is a place outside of the interval with a horizontal tangent line. The instructor clarified that the task was only to attend to whether or not the property held on the interval from \(a\) to \(b\), erased the portion of the graph outside of the interval (shown in Fig. 10), and the student who initially offered the critique agreed that Fig. 10b was now acceptable.

However, another student questioned Fig. 10b because he recognized that even though the function was not defined at one point, it looked like the limit would still exist. At this point, the instructor led the class in a lengthy discussion about the definition of the derivative and why tangent lines do not exist at places where there is a removable discontinuity. No other class went into this much...
depth about the definition of a tangent line.

When no other student offered any critiques of the examples in Fig. 9, Instructor A pointed out a problem with graph 9b and warned the students to be careful when drawing their graphs. She then modified the graph. The instructor then led the class in a discussion about what the graphs on the board had in common. First, she highlighted the five graphs that had some sort of discontinuity and asked the students what the other five graphs (9b, 9e, 9f, 9i, 9j) had in common. At least one student responded that those graphs had a point or a cusp, and Instructor A introduced the term differentiable and emphasized that all of the graphs that were drawn without a tangent line parallel to the secant line were either not continuous or not differentiable.

Instructor A: So, we can still draw some continuous graphs that don’t have this property. This one, and this one, and this one. So, these three at least are continuous but don’t have the property. What’s special about those guys?

Students (two): Pointy.
Instructor A: They’ve got pointy places, or cusps, or peaks or something like that. What’s that mean?
Student: It’s undefined
Instructor A: What’s undefined?
Students: (many responses, incomprehensible)
Instructor A: The slope, the derivative. So, the derivative is not defined everywhere on this graph. So, this graph, we say is not differentiable. I don’t know that I’ve used that terminology yet. So, not differentiable means that the derivative does not exist somewhere.

Then, the instructor gave the class a short period of time to think about graphs that are both continuous and differentiable on the interval to decide if those graphs had to have a place where the tangent and secant lines were parallel.

The full set of examples that had been generated and discussed culminated with the instructor writing the language of the MVT on the board and relating it to what the students had created. For example, when stating that the function must be continuous on the closed interval [a, b], she referred to the example in Fig. 9d to illustrate that an open interval would not have guaranteed that the property held. She also referred back to the discussion about the definition of the derivative to explain why the function need only be differentiable on an open interval (a, b).

Overall, we use this vignette of Instructor A’s class to highlight the way in which examples can be generated from both the instructor and students, can elicit responses from both students and the instructor (to address imprecisions and to highlight key mathematical ideas), and can be used to build mathematical concepts in robust ways. Therefore, this serves as one way in which the four themes we highlight in this work come together around each example individually and as a collection. By considering how examples were generated and used in this classroom, we can see ways in which students were given opportunities to engage in the mathematical work of the classroom. As a result of this particular instruction, we believe that students had opportunities to reason about and make connections among mathematical examples, develop a sense of the way in which examples serve a collective effort to build mathematical ideas, and develop a sense of themselves as individuals who can reason about mathematics.

5.2. Classroom vignette: Instructor E

Instructor E began instruction on the MVT by inviting students to join her in “playing a game.” She told students they would set up a function and drew two axes, announcing that she will draw a curve, but not yet. She told the class that they would look at the function on the interval between a and b and marked these on the x-axis. Next, she introduced the “rules” of the game, writing each of these using orange marker on the whiteboard. She told the students that the function is continuous on the interval (a, b), f is differentiable on (a, b), and that \( f(a) = f(b) \). With this, she then added points \( (a, f(a)) \) and \( (b, f(b)) \) to the graph such that \( f(a) = f(b) \), gesturing to emphasize that these were directly across from each other (see Fig. 11). Instructor E asked students to tell her, before she drew the curve, what is the slope of the secant line between the two points. A student responded “zero” and Instructor E confirmed, adding a fourth rule on the board in orange: “slope of secant line = 0.”

Instructor E then announced, “Okay, here’s the game…I’m going to draw the curve. And there are an infinite number of curves I can draw between those two points.” She then asked the class: “So for any curve I draw, true or false, somewhere on that curve the tangent line slope is equal to zero.” Many students quickly responded true, which she acknowledged. Instructor E then elaborated on what this curve would look like and why that would result in a secant line with a slope of zero. As she spoke, she gestured with her hands to indicate that if the graph goes up, it must come back down, and if it goes down, it must come back up. She did not address the possibility of a function that was a horizontal line, and at this point, she had not yet drawn a curve on the axes. Instructor E then
asked students if it could happen more than once (i.e., that there could be more than one point on the curve at which the slope of the tangent line is zero) and confirmed that it could happen many times. Following this, she wrote on the board in black: “Somewhere on the curve, the tangent line’s slope is also zero.” Instructor E emphasized to students that this is Rolle’s Theorem, reiterating that if all of the statements in orange (i.e., the “rules”) are true, the statement in black must also be true.

Instructor E next erased the first two conditions, asking students if the final statement (the slope of the tangent line must somewhere be zero) is now true. Several students said no, and one student asked a question that was not audible on the video. (Based on the instruction that follows, we believe the question was about graphs with corners or cusps.) Instructor E recognized the several “no votes” and then tabled the additional question, asking the student if he could bring that up later. She then stated, “I took the continuous part out, so how’s this?” and drew a graph on the axes (Fig. 12). Note that this was the first graphical example that was shown during Instructor E’s lesson. She reiterated that the theorem “fails” since the graph does not have a “leveling off,” and stated, “You need all of the orange things…. There (gesturing to the graph) is a counterexample.”

Next, Instructor E rewrote the first condition (that \( f \) is continuous), again leaving out the second (that \( f \) is differentiable). She returned to the student who had posed the earlier question and interpreted that contribution as the graph in Fig. 13, pointing out that it is still continuous but not differentiable, and there is no leveling off. At this point, Instructor E modified the example marking a more distinct corner, thus attending to the precision of her example. She again asserted that all four conditions are needed (i.e., the rules that were written in orange) in order to conclude that there is a point on the curve where the tangent has a slope of zero.

As she erased the curve, Instructor E said, “Please, please, please, don’t be cute” on the exam. She told students that if you are asked on the exam to provide an example of a function that is continuous but “fails Rolle’s Theorem,” draw a simple one. She pointed out that students in the past have been “cutesy,” drawing a graph (Fig. 7, discussed in Section 4.5) with many points at which the function is not differentiable but that also has points at which the tangent has a slope of zero, meaning the student got the exam question marked wrong. Instructor E recommended that if they “want to fail a math theorem, always fail it simply,” drawing another example (Fig. 7). While this can be sound advice regarding common mathematical practices, the context of this recommendation was in service of avoiding common pitfalls on an exam.

She then erased the example and drew a graph that met all of the stated conditions (Fig. 14), pointing out that points marked with \( c \) are used to designate where the theorem “does work,” gesturing on the board where the horizontal tangent line would be.

Under the premise of telling a “similar story,” the instructor erased the third condition— \( f(a) = f(b) \) —asserting that \( f(a) \) does not need to equal \( f(b) \) and the fourth condition, that the slope of the secant line is zero. She returned to the graph in Fig. 14 and asked students to imagine keeping the point \((a, f(a))\) where it is and rotating the point \((b, f(b))\) up or down, gesturing up and down as she spoke. She drew a new set of axes that depicted \( f(b) \) greater than \( f(a) \) and rewrote the two conditions, “continuous on \([a, b]\)” and
“differentiable on \((a, b)\).” After writing a modified version of the true or false statement from before, the students were asked if the theorem would still be true in this new situation. Without waiting for a response, Instructor E returned to the graph in Fig. 14 and, again, asked students to imagine rotating it (gesturing with her hands), concluding that the conclusion is still true. Instructor E drew a curve to illustrate this new set of conditions (Fig. 15) and affirmed that there was a value \(c\) on the curve where the slope of the tangent line equals the slope of the secant line. After asking a series of rhetorical questions about the validity of this idea and affirming each one, she added to the graph, creating the full picture as seen in Fig. 15. She asked students to think about rotating point \((b, f(b))\) down and compared it to the graph in Fig. 15. After a student asked if this was still Rolle’s Theorem, Instructor E responded that it is not and wrote “Mean Value Theorem” on the board.

Instructor E drew attention to the fact that there could be “more than one \(c\)” and went on to describe a curve that goes “up, down, and then up” having two such points, and one that goes “up, down, up, and down” having three, gesturing as she spoke, though not sketching these examples.

The lesson moved on to an exercise, “Verify the MVT for \(f(x) = x^3 + 2x^2 + x\) on \([-1, 2]\).” Notable with this exercise is that the instructor did not sketch what the curve might look like on that interval, pointing out that students can “imagine” the curve on the given interval and can tell that it is continuous and differentiable without needing to draw it, though she made that claim with no
evidence in-the-moment. She proceeded to work through the example, concluding by giving the students a second problem to work on their own, pointing out that the problems asked them to “verify,” meaning the MVT is true and they just need to find the c values.

This vignette provides a contrast to that of Instructor A’s classroom. Students generated none of the examples that were discussed in Instructor E’s class. In general, students had very little voice in the lesson, despite the rhetorical valuing from the instructor that students “understand” the ideas being presented. Instructor E’s lesson illustrates one in which all of the examples discussed were used as “existence proofs,” often the only one, aside from comments made by the instructor about other possible cases. Many of the examples were presented as ones that may be useful on exam questions, because of their simplicity in addressing a given set of conditions. In this vignette, we see students positioned as passive observers of the mathematical work being conducted by the instructor. While the lesson featured teacher questioning, we believe the focus on the generation and use of examples highlights that this pedagogical feature did not position students with opportunities to engage in mathematical practices or contribute their thinking about key ideas.

6. Discussion

In this paper, we have highlighted our work investigating the way examples were generated and used in instruction focused on defining and illustrating the MVT. It should be noted that our work focused on instruction of the MVT, and our themes were derived from that lesson. It is certainly possible that additional themes would emerge when evaluating lessons on different mathematical topics. These findings still provide an opportunity to consider the role of the generation and use of examples in supporting student engagement in an active learning environment, drawing on a social theory of learning (Wenger, 1998) as a frame for defining learning and engagement. In the sections below, we discuss the connection between these themes and student engagement, as well as the mathematical richness of the respective lessons. We offer implications for teaching and research.

6.1. Connecting the generation and use of examples to student engagement

The four themes we detailed and illustrated using the two vignettes above serve as ways to characterize how examples are generated and used in mathematics classrooms and how (or if) opportunities are created for students to be engaged in developing an understanding of a mathematical topic. However, these themes, and the variations within each, do not provide a prescription. For example, the takeaway is not that learner-generated examples are best in all circumstances, or that instruction should not include instructor-generated examples at all. Both have their place in high-quality lessons, in part determined by how these contributions are taken up and used as part of the construction of mathematical ideas. In the sections below, we revisit our four themes and view each through the lens of a social theory of learning to consider the implications of how examples are generated and used in mathematics classrooms on student engagement and learning.

6.1.1. Who generated the example?

The three variations of who generated an example serve as one way to consider the opportunities students have to engage with and learn mathematics. Particularly illuminating are the instructor-interpreted examples, which recognize that students can be seen as participants in the mathematics activity of the classroom, though often only through short responses to questions designed to elicit specific ideas (e.g., “Initiation-Response-Evaluation” questioning patterns; Mehan, 1979). In these cases, the instructor is still the sole individual positioned to have the opportunity to make her sense-making public and actively engage in mathematics practices. While students may be making meaning individually, in the collective space of the classroom they are positioned as observers of this work, with an occasional opportunity for minor contributions that ultimately serve the work being done by the instructor. This can impact the way in which students develop an identity regarding their role in the mathematics classroom and as doers of mathematics (Aguirre, Mayfield-Ingram, & Martin, 2013; Gresalfi, Martin, Hand, & Greeno, 2009). This suggests that understanding the nature of quality of students’ participation in the classroom goes beyond quantifying the number of questions posed to students and the number of student contributions. In the case of attending to how examples are generated, students need at least some opportunities to offer complete ideas. This is supported pedagogically by providing students with substantive time and prompts to construct and contribute mathematical reasoning. Notable, though, is that these opportunities for learner-generated examples serve as a necessary, though not sufficient, feature of an active learning environment.

6.1.2. How precise was the example?

Contributions that are incorrect or incomplete are a part of classroom discussion and are, furthermore, part of the work of doing and learning mathematics (Brodie, 2014; Nesher, 1987). Even when these contributions emerge from an incomplete conception of an idea, they provide a window into an individual’s reasoning, which serves as useful data for discussion. This points to the importance of allowing students to share their reasoning publicly, such as through generating and responding to examples.

The takeaway here is not that these less-than-precise contributions should be avoided, or that they can be avoided. From the standpoint of considering students’ positions within the mathematics classroom community, moves to disregard, quickly resolve, or altogether silence ideas that are not precise marginalize students’ participation in mathematical activity. This impacts their developing identity as doers of mathematics and, potentially, their opportunities to learn and make meaning of mathematics. The nuance regarding the precision of a contribution that emerged from our analyses—not just that something is explicitly correct or incorrect—is important for understanding and appreciating the ideas that are shared in a mathematics classroom. By identifying the
precision of an example as a characteristic, it becomes part of the important consideration of who responds to those ideas, how those ideas are responded to and evaluated, and how and when those ideas are drawn upon, regardless of precision.

6.1.3. Who responded to the example?

In our data, we see that the instructor was often positioned as the primary individual in the classroom to generate or interpret examples. In our five lessons, the limited instances of students responding to examples primarily occurred when the examples were generated by the students. Students’ input on examples generated or interpreted by the instructor was restricted to individual questions or wonderings. These responses came from the position of figuring out why the example was correct, often without any explicit discussion, as opposed to determining if the example was correct. In some cases, these contributions were not even taken up by the instructor. In contrast, when their peers provided the examples—although infrequently—the students seemed to be more readily put in position to respond to and critique the examples.

A shift in the classroom culture that encourages students to be evaluating all examples could promote student engagement and learning. Students could be encouraged to evaluate examples in a way similar to what was seen in Instructor A’s class, by providing several examples and giving students time to evaluate them for specific characteristics and make conjectures about properties that might hold. To support this, instructors can create prompts that allow students to generate many examples and explore the ways in which the examples change as conditions change (Mason & Watson, 2008).

We noticed instances of examples in our data (often generated or interpreted by the instructor) that did not face any response at all before the instructor moved on to new ideas. While there might be reasons to decide to move on from an idea that has not been resolved or about which there are still questions or uncertainty, we did not interpret those intentions from our data. It is expected that when an instructor decides to “table” an idea the idea will come up later in the lesson as the instructor makes a desired point or connection. In our data, when the instructor moved on from an idea, it was not revisited. This type of tabling of ideas also happened when students did provide a response or ask a question. For example, the student who proposed the idea of an asymptote in Instructor D’s class but got a hole discontinuity instead.

As with the way imprecise ideas are handled in the classroom, when students’ contributions are disregarded or cast aside, opportunities to participate and learn are restricted. Furthermore, responding to mathematical ideas—asking questions, providing critiques, pressing for more information—is central to mathematical activity. Our findings highlight the limited opportunities for students to engage in such practices.

6.1.4. How was the example used?

The final theme that emerged from our analyses highlighted the variety of ways in which an example could be used. This variety represented a legitimate set of ways examples could be used in a mathematics classroom and as part of developing mathematical ideas, particularly with topics like the MVT. Through this theme, we highlight that examples can take different forms and be used for different purposes. Furthermore, engaging students more authentically in the mathematical work of the classroom would entail providing them opportunities to participate in and contribute to the full range of uses of examples. For example, if students’ opportunities to generate and evaluate examples only come in the context of one purpose, or at a particular time in a lesson or sequence of lessons, then students are positioned as limited contributors to the mathematical work of the classroom. This can have negative impacts on their developing identities as individuals who can contribute to developing mathematical ideas. This theme adds that instructors need to not only consider who generates and responds to examples, but also how the examples are being used.

6.2. Connecting student engagement and the depth of the mathematics

While the four themes tend to emphasize the nature of student engagement in a classroom, we certainly want to emphasize that the content of the lesson cannot be ignored. We stress the importance of students having the opportunity to explore mathematics at an appropriate level of mathematical depth. One could certainly imagine a lesson where students generate and respond to a variety of examples without exploring the mathematics with the appropriate depth. By depth, we mean addressing all the salient features of the MVT and expecting students to understand what these features mean and why they are necessary. This contrasts with merely having students solve routine problems or only develop a surface-level understanding of the mathematical concept.

We found the mathematics to be deeper in Instructor A’s class than in any of the other classes in our study. Incidentally, this occurred in about the same amount of class time as the other four lessons and included significantly more opportunity for student engagement. An example of this depth in the mathematics content is when Instructor A’s class discussed the need for the condition that the function be continuous on a closed interval $[a, b]$ as opposed to an open interval. Instructor A used the learner-generated example in Fig. 9c to illustrate this important detail in the MVT. No other lesson matched this level of depth and precision to address this detail in the theorem. Instructor B had an instructor-generated example on the board that was extremely similar to Fig. 9c, but she did not discuss why this example illustrated the need for the function to be continuous on a closed interval. Instructor C and E did not address the need for a closed interval at any point during their lessons. Furthermore, Instructor D, claimed (incorrectly) that a closed interval was required in the theorem so that it would be possible to compute the average rate of change. We find it unlikely that students will be able to make critical connections such as those highlighted in Instructor A’s class discussion on their own, and believe that it is ultimately the instructor’s responsibility to make sure that such connections are made obvious to the students, in part through a strategic use of examples.

Additionally, in Instructor A’s class with learner-generated examples, there was more diversity in the examples. These learner-generated examples included graphs with removable discontinuities, jumps, vertical asymptotes, and cusps. The instructor-generated
examples in the other classes did not exhibit this level of diversity in examples. Moreover, the students in Instructor A’s class showed evidence of evaluating examples to understand the material at a deeper level. As discussed previously, students engaged in a discussion of the example in Fig. 9a about the relevance of the definition of a derivative and the definition of a tangent line with respect to the MVT.

While it is often hypothesized that active learning takes more class time and instructors fear that they will not be able to cover the same amount of material if students are given the opportunity to participate more fully during a class, our data provides evidence that this is not always the case. Class sessions introducing the MVT ranged from 38 to 53 minutes, with Instructor A’s class spending 42 minutes on the topic. The examples that emerged during the lessons from Instructors B, C, D, and E seemed to be pre-determined, primarily generated or interpreted by the instructor, and designed to move the lesson along a pre-determined trajectory. With Instructors D and E, we saw cases where students’ questions were ignored or at best delayed if the instructor didn’t find it to be the right time to address an issue. These heavily directed approaches still took a full class session, with some time left to allow for students to work on practice problems.

7. Conclusion

In a classroom that facilitates students’ mathematical learning as we have framed it in this paper, the instructor is tasked with supporting students in engaging in mathematical practices in the context of the classroom community. This has implications for the way in which the instructor represents mathematics (for example, the role of examples in developing mathematical ideas) as well as how the instructor engages students in that effort. For instance, in addition to considering who generates examples, how they are generated, and who and how they are responded to, the instructor needs to also be skilled in leading a discussion about the examples in a way that moves the lesson forward toward particular goals. Further, he or she needs to know which examples to highlight in order to demonstrate concepts and provide an appropriate amount of breadth and depth in a lesson.

We emphasize that this does not simply mean that students should have more opportunity to work in groups or that students should talk more during class; instead, we argue that the nature of students’ engagement in mathematical activity is critical for their learning of key mathematical concepts. Our focus on the generation and use of examples contributes to a sense of what is entailed in students’ active learning in mathematics. These findings have implications for how instructors can be supported—through materials, coordination, or instructional support—to create classroom environments that actively engage students in doing mathematics.

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References
