$\ell_2$ and $\ell_1$ Regularization of Ill-conditioned Problems

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Example: diffusion equation

\[ u_t = u_{xx}, \quad (t, x) \in (0, \infty) \times (-1, 1) \]

\[ u(t, -1) = u(t, 1) = 0, \quad u(0, x) = u_0(x) \]

Forward problem: given \( u_0 \) find \( u(t, x) \) for some \( t > 0 \).

Backward problem: given \( u(T, x) \) find \( u_0 \).
Blurring/deblurring of images

original

blurred
Tikhonov regularization

\[ J_\lambda(x) = \| b - Ax \|^2 + \lambda \| x \|^2, \]

or more generally

\[ J_{B,\lambda}(x) = \| b - Ax \|^2 + \lambda \| Bx \|^2. \]

\( B \) can be a differential operator and \( \| Bx \| \) an approximation to a Sobolev norm.

Numerically, we minimize

\[ J_{B,\lambda}(x) = \| b - Ax \|_2^2 + \lambda \| Bx \|_2^2. \]

by solving the concatenated least-squares problem,

\[ J_{B,\lambda}(x) = \left\| \begin{bmatrix} A & 0 \\ \sqrt{\lambda}B \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2^2. \]
Numerical example: diffusion equation

\[ u_t = u_{xx} \] with Neumann BCs.
In discretized form,

\[ U_k = M^k U_0. \]

Tikhonov regularization: minimize \( \|M^K u - \tilde{U}_k\|^2 + \lambda^2 \|u\|^2 \)
$u_t = u_{xx}$ with Neumann BCs.
In discretized form,

$$U_k = M^k U_0.$$ 

Tikhonov regularization: minimize $\|M^k u - \tilde{U}_k\|^2 + \lambda^2 \|Du\|^2$
Numerical example: diffusion equation

\( u_t = u_{xx} \) with Neumann BCs. In discretized form,

\[ U_k = M^k U_0. \]

Tikhonov regularization: minimize

\[ \| M^k u - \tilde{U}_k \|^2 + \lambda^2 \| D_2 u \|^2 \]
Numerical example: piecewise constant case

\[ u_t = u_{xx} \] with Neumann BCs. In discretized form,

\[ U_k = M^k U_0. \]

\( \ell_1 \) v.s. \( \ell_2 \) regularization:

\[ \| Du \|_2 \] vs. \[ \| Du \|_1 (TV) \]
\( \ell_1 \) regularization, sparsity, and compressive sensing

Minimize \( \|x\|_{\ell_1} \) subject to \( Ax = b \)

Minimize \( \|x\|_{\ell_1} \) subject to \( \|Ax - b\|_2 \leq \sigma \)

Minimize \( \|Ax - b\|^2 + \lambda \|x\|_{\ell_1} \)

Minimize \( \|Ax - b\|^2 + \lambda \|Bx\|_{\ell_1} \)

Remark: In statistics, Lasso regression analysis.
Underdetermined systems and compressive sensing

Solve

\[ Ax = b, \]

where \( A \) is \( m \times N \) and \( m < N \).
Underdetermined systems and compressive sensing

Solve

\[ Ax = b, \]

where \( A \) is \( m \times N \) and \( m < N \).

In CS, the goal is to obtain sparse solutions, i.e., \( x_j \approx 0 \), for several \( j \)'s.
Underdetermined systems and compressive sensing

\[ \begin{bmatrix} 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ x_0 & x_1 & \cdots & x_N \end{bmatrix} \]

Solve

\[ Ax = b, \]

where \( A \) is \( m \times N \) and \( m < N \).

In CS, the goal is to obtain sparse solutions, i.e., \( x_j \approx 0 \), for several \( j \)'s.

One option: Minimize \( \| x \|_{\ell_1} \) subject to \( Ax = b \).

\[ \| x \|_{\ell_p} = (|x_0|^p + |x_2|^p + \cdots + |x_N|^p)^{1/p} \]

Why \( p = 1 \)?

Remark: the location of nonzero \( x_j \)'s is not known in advance.
Why $\ell_1$?

Unit ball:
$\ell_0$, $\ell_{1/2}$, $\ell_1$, $\ell_2$, $\ell_4$, $\ell_\infty$

$\|x\|_{\ell_p} = (|x_0|^p + \cdots + |x_N|^p)^{1/p}$

or, for $0 \leq p < 1$,

$\|x\|_{\ell_p} = (|x_0|^p + \cdots + |x_N|^p)$

- $\|x\|_{\ell_0} = \#$ of nonzero entries in $x$
  ideal (?) but leads to a NP-complete problem
- $\ell_2$ computationally easy but does not lead to sparse solutions.
Sparsity and the $\ell_1$-norm (1 equation case)

Example

\[ a_1 x_1 + a_2 x_2 = b_1 \]
Sparsity and the $\ell_1$-norm (1 equation case)

Example – $\ell_2$

$$\min_{x_1, x_2} \sqrt{x_1^2 + x_2^2} \quad \text{subject to} \quad a_1 x_1 + a_2 x_2 = b_1$$
Sparsity and the $\ell_1$-norm (1 equation case)

Example – $\ell_1$

$$\min_{x_1, x_2} |x_1| + |x_2| \quad \text{subject to} \quad a_1 x_1 + a_2 x_2 = b_1$$
See matlab experiment! (Test-l1-l2.m)
Back to image deblurring – TV reconstruction
Back to image deblurring – HOTV order 2
Back to image deblurring – HOTV order 3
\( \ell_2 \) vs. \( \ell_1 \) in image deblurring (1D slice)

\( \ell_2 \) (black) and \( \ell_1 \) (green) reconstructions