ℓ_2 and ℓ_1 Regularization of Ill-conditioned Problems

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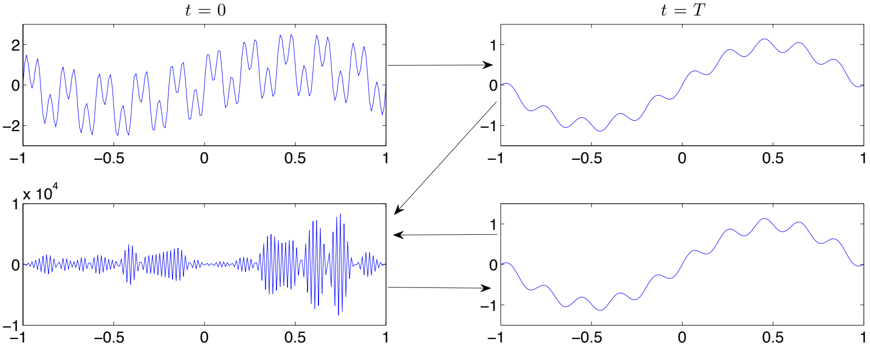
Example: diffusion equation

$$u_t = u_{xx}, \quad (t,x) \in (0,\infty) \times (-1,1)$$

$$u(t,-1) = u(t,1) = 0, \quad u(0,x) = u_0(x)$$

Forward problem: given u_0 find $u(t,x)$ for some $t > 0$.

Backward problem: given $u(T,x)$ find u_0.



Blurring/deblurring of images

original

blurred

Tikhonov regularization

$$J_\lambda(x) = \|b - Ax\|^2 + \lambda\|x\|^2,$$

or more generally

$$J_{B,\lambda}(x) = \|b - Ax\|^2 + \lambda\|Bx\|^2.$$

B can be a differential operator and $\|Bx\|$ an approximation to a Sobolev norm.

Numerically, we minimize

$$J_{B,\lambda}(x) = \|b - Ax\|_2^2 + \lambda\|Bx\|_2^2.$$

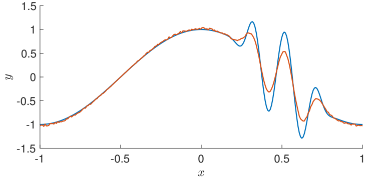
by solving the concatenated least-squares problem,

$$J_{B,\lambda}(x) = \left\| \begin{bmatrix} A \\ \sqrt{\lambda}B \end{bmatrix} x - \begin{bmatrix} b \\ 0 \end{bmatrix} \right\|_2^2.$$

Numerical example: diffusion equation

$u_t = u_{xx}$ with Neumann BCs.
In discretized form,

$$U_k = M^k U_0.$$



Tikhonov regularization: minimize $\|M^K u - \tilde{U}_k\|^2 + \lambda^2\|u\|^2$

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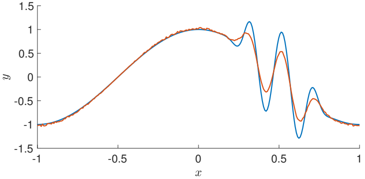

Tikhonov regularization: minimize $\|M^K u - \tilde{U}_k\|^2 + \lambda^2 \|Du\|^2$

Numerical example: diffusion equation

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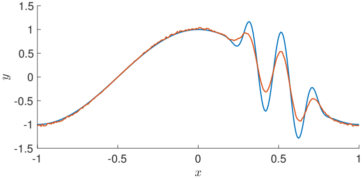


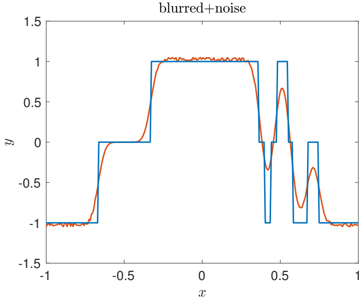
Tikhonov regularization: minimize $\|M^K u - \tilde{U}_k\|^2 + \lambda^2 \|D_2 u\|^2$

Numerical example: piecewise constant case

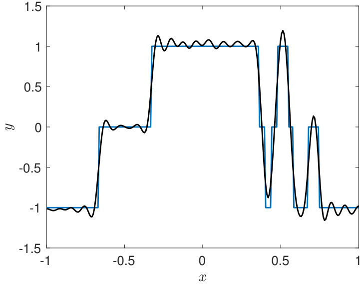
$u_t = u_{xx}$ with Neumann BCs.
In discretized form,

$$U_k = M^k U_0.$$

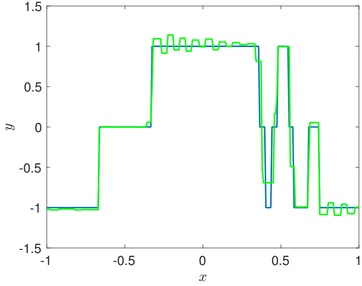
ℓ_1 v.s. ℓ_2 regularization:



$\|Du\|_2$



$\|Du\|_1(TV)$



ℓ_1 regularization, sparsity, and compressive sensing

Minimize $\|x\|_{\ell_1}$ subject to $Ax = b$

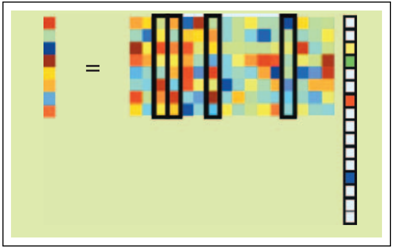
Minimize $\|x\|_{\ell_1}$ subject to $\|Ax - b\|_2 \leq \sigma$

Minimize $\|Ax - b\|^2 + \lambda\|x\|_{\ell_1}$

Minimize $\|Ax - b\|^2 + \lambda\|Bx\|_{\ell_1}$

Remark: In statistics, Lasso regression analysis.

Underdetermined systems and compressive sensing

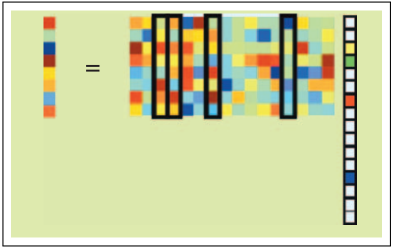


Solve

$$Ax = b,$$

where A is $m \times N$ and $m < N$.

Underdetermined systems and compressive sensing



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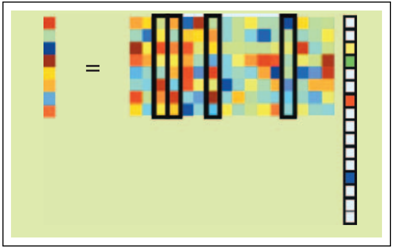
In CS, the goal is to obtain sparse solutions, i.e., $x_j \approx 0$, for several $j's$.

Underdetermined systems and compressive sensing



Solve

$$Ax = b,$$

where A is $m \times N$ and $m < N$.

In CS, the goal is to obtain sparse solutions, i.e., $x_j \approx 0$, for several $j's$.

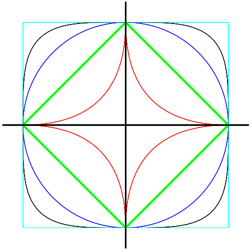
One option: Minimize $\|x\|_{\ell_1}$ subject to $Ax = b$.

$$\|x\|_{\ell_p} = \left(|x_0|^p + |x_2|^p + \cdots + |x_N|^p\right)^{1/p}$$

Why $p = 1$?

Remark: the location of nonzero x_j's is not known in advance.

Why ℓ_1?



Unit ball:
ℓ_0, $\ell_{1/2}$, ℓ_1, ℓ_2, ℓ_4, ℓ_∞

$$\|x\|_{\ell_p} = \left(|x_0|^p + \cdots + |x_N|^p\right)^{1/p}$$

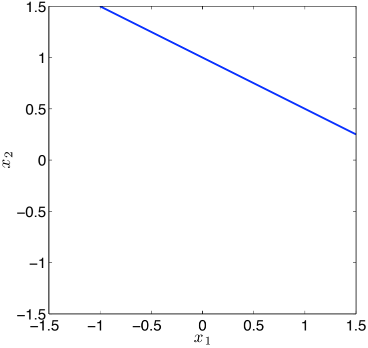
or, for $0 \leq p < 1$,

$$\|x\|_{\ell_p} = \left(|x_0|^p + \cdots + |x_N|^p\right)$$

- $\|x\|_{\ell_0} = \#$ of nonzero entries in x
  ideal (?) but leads to a NP-complete problem
- ℓ_2 computationally easy but does not lead to sparse solutions.

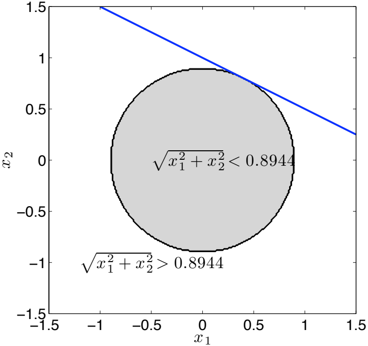
Example

$$a_1 x_1 + a_2 x_2 = b_1$$



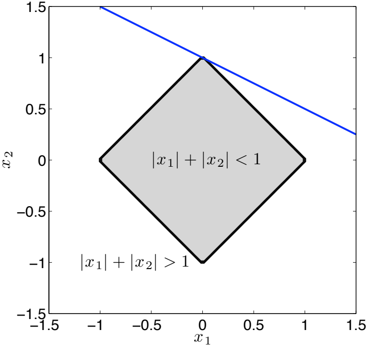
Example – ℓ_2

$$\min_{x_1, x_2} \sqrt{x_1^2 + x_2^2} \quad \text{subject to} \quad a_1 x_1 + a_2 x_2 = b_1$$



Example – ℓ_1

$$\min_{x_1,x_2} |x_1| + |x_2| \quad \text{subject to} \quad a_1 x_1 + a_2 x_2 = b_1$$



See matlab experiment! (Test-l1-l2.m)

Back to image deblurring

original

blurred+noise

Back to image deblurring – TV reconstruction

original

deblurred

Back to image deblurring – HOTV order 2

original

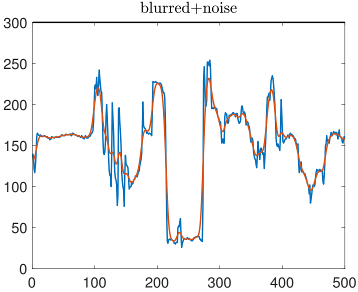
deblurred

Back to image deblurring – HOTV order 3

original

deblurred

ℓ_2 vs. ℓ_1 in image deblurring (1D slice)



ℓ_2 (black) and ℓ_1 (green) reconstructions