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APM 503 Master's Portfolio Project

Properties of Inner Product Spaces

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In this project, I supplement the coursework from APM 503, Applied Analysis, with additional work on inner product spaces, which were discussed in the supplementary material in Professor Horst Thieme's lecture notes, but were not covered in class. First, I will copy, without proof, a few important definitions and theorems from those lecture notes, then I will solve several exercises.

A.1 Definition. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. An *inner product* on a vector space X over $\mathbb{K}$ is a function $(u, v) \mapsto \langle u, v\rangle$ from $X \times X$ into $\mathbb{K}$ with the following properties for all $u, v, w \in X$, $\alpha \in \mathbb{K}$:

(i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$, which implies that $\langle u, u\rangle$ is real.

(ii) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$. (associative law)

(iii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$. (distributive law)

(iv) $\langle u, u\rangle > 0$ if $u \neq \mathbb{O}$.

These properties immediately lead to the following:

(v) $\langle u, \alpha v\rangle = \overline{\alpha}\langle u, v\rangle$ and $\langle \alpha u, \alpha v\rangle = |\alpha|^2\langle u, v\rangle$, by (i) and (ii).

(vi) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$, by (i) and (iii).

(vii) $\langle \mathbb{O}, u\rangle = \langle 0\mathbb{O}, u\rangle = 0\langle \mathbb{O}, u\rangle = 0$

(viii) $\langle u, \mathbb{O}\rangle = \overline{\langle \mathbb{O}, u\rangle} = 0$

A.2 Theorem (the Cauchy-Schwartz inequality). Let X be an inner product space. If $u, v \in X$, then

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle$$

with equality if and only if u and v are linearly dependent. Additionally, we define the norm induced by the inner product

$$\|u\| = \sqrt{\langle u, u\rangle}$$

and with this notation, the Cauchy-Schwartz inequality takes the form

$$\langle u, v\rangle \leq \|u\|\,\|v\|\,.$$

A.4 Corollary. If $\langle , \rangle$ is an inner product on a vector space X, then $\|\cdot\|$ given by $\|u\| = \sqrt{\langle u, u\rangle}$ is a norm on X.

A.8 Theorem (the parallelogram law). A normed vector space X is an inner product space if and only if its norm satisfies the parallelogram law

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \forall u, v \in X.$$

A.9 Proposition. Let M be a complete convex set in an inner product space X and $u \in X$. Then there exists a unique $v \in M$ such that $d(u, M) = \|u - v\|$.

Exercises

A.1.1) Let X be a vector space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$.

a. $\langle u, v\rangle + \langle v, u\rangle = \frac{1}{2}(\|u+v\|^2 - \|u-v\|^2)$.

Proof. By the properties of inner products in definition A.1 and the definition of the norm in equation A.1:

$$\begin{aligned}
&\langle u, v\rangle + \langle v, u\rangle \\
&= \frac{1}{2}\left[2\langle u, v\rangle + 2\langle v, u\rangle\right] \\
&= \frac{1}{2}\left[\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle - \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, v\rangle)\right] \\
&= \frac{1}{2}\left[\langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle + \langle v, v\rangle - (\langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle + \langle v, v\rangle)\right] \\
&= \frac{1}{2}\left[\langle u+v, u+v\rangle - \langle u-v, u-v\rangle\right] \\
&= \frac{1}{2}\left[\|u+v\|^2 - \|u-v\|^2\right]
\end{aligned}$$

□

b. In a real inner product space, $\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)$.

Proof. In a real inner product space, $\langle u, v\rangle = \langle v, u\rangle$ so by part a:

$$2\langle u, v\rangle = \langle u, v\rangle + \langle v, u\rangle = \frac{1}{2}(\|u+v\|^2 - \|u-v\|^2)$$

Thus:

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2).$$

□

c. If X is a complex inner product space,

$$\langle u, v\rangle - \langle v, u\rangle = \frac{i}{2}(\|u + iv\|^2 - \|u - iv\|^2)$$

and

$$\langle u, v\rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 + i\,\|u + iv\|^2 - i\,\|u - iv\|^2\right).$$

Proof. If X is a complex inner product space:

$$\langle u, v\rangle - \langle v, u\rangle = \frac{i}{2}\left[2i\langle v, u\rangle - 2i\langle u, v\rangle\right]$$

$$= \frac{i}{2}\left[\langle u, u\rangle + i\langle v, u\rangle - i\langle u, v\rangle + \langle v, v\rangle - \langle u, u\rangle + i\langle v, u\rangle - i\langle u, v\rangle - \langle v, v\rangle\right]$$

$$= \frac{i}{2}\left[\langle u, u\rangle + i\langle v, u\rangle - i\langle u, v\rangle + \langle v, v\rangle - (\langle u, u\rangle - i\langle v, u\rangle + i\langle u, v\rangle + \langle v, v\rangle)\right]$$

$$= \frac{i}{2}\left[\langle u, u\rangle + \langle iv, u\rangle + \langle u, iv\rangle + \langle iv, iv\rangle - (\langle u, u\rangle + \langle -iv, u\rangle + \langle u, -iv\rangle + \langle -iv, -iv\rangle)\right]$$

$$= \frac{i}{2}\left[\langle u + iv, u + iv\rangle - \langle u - iv, u - iv\rangle\right]$$

$$= \frac{i}{2}\left[\|u + iv\|^2 - \|u - iv\|^2\right] \text{ by definition of the norm.}$$

Then:

$$\langle u, v\rangle = i\langle u, iv\rangle = \frac{i}{2}[\langle u, iv\rangle - \langle iv, u\rangle + \langle iv, u\rangle + \langle u, iv\rangle]$$

$$= \frac{i}{2}[\langle u, iv\rangle - \langle iv, u\rangle + i\langle v, u\rangle - i\langle u, v\rangle]$$

$$= \frac{i}{2}[(\langle u, iv\rangle - \langle iv, u\rangle) - i(-\langle v, u\rangle + \langle u, v\rangle)]$$

$$= \frac{i}{2}\left[\frac{i}{2}(\|u + iiv\|^2 - \|u - iiv\|^2) - i\left(\frac{i}{2}(\|u + iv\|^2 - \|u - iv\|^2)\right)\right]$$

$$= \frac{-1}{4}\left[(\|u - v\|^2 - \|u + v\|^2) - i\left(\|u + iv\|^2 - \|u - iv\|^2\right)\right]$$

$$= \frac{1}{4}\left[(-\|u - v\|^2 + \|u + v\|^2) + i(\|u + iv\|^2 - \|u - iv\|^2)\right]$$

$$= \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2 + i\,\|u + iv\|^2 - i\,\|u - iv\|^2\right).$$

□

A.1.2a) A real $n\times n$ matrix $A=(\alpha_{ij})$ is symmetric (meaning $\alpha_{ij}=\alpha_{ji}$ for all $i,j\in\{1,...,n\}$) if and only if $x\cdot(Ay)=(Ax)\cdot y$ for all $x,y\in\mathbb{R}^n$.

Proof. Let $x,y\in\mathbb{R}^n$, let $A=(\alpha_{ij})$ be a real $n\times n$ matrix, and let a subscript i on a vector denote the i^{th} component of that vector. Then:

$$x\cdot(Ay)=\Sigma_{i=1}^n x_i(Ay)_i=\Sigma_{i=1}^n x_i\Sigma_{j=1}^n\alpha_{ij}y_j=\Sigma_{i=1}^n\Sigma_{j=1}^n x_i\alpha_{ij}y_j$$

$$(Ax)\cdot y=\Sigma_{j=1}^n(Ax)_j y_j=\Sigma_{j=1}^n y_j\Sigma_{i=1}^n\alpha_{ji}x_i=\Sigma_{j=1}^n\Sigma_{i=1}^n y_j\alpha_{ji}x_i$$

$\Rightarrow$ If A is symmetric, then trivially for all $x,y\in\mathbb{R}^n$:

$$x\cdot(Ay)=\Sigma_{i=1}^n\Sigma_{j=1}^n x_i\alpha_{ij}y_j=\Sigma_{j=1}^n\Sigma_{i=1}^n y_j\alpha_{ji}x_i=(Ax)\cdot y$$

$\Leftarrow$ If $x\cdot(Ay)=(Ax)\cdot y$ for all $x,y\in\mathbb{R}^n$, then let $i,j\in\{1,...,n\}$, let x be e^i (the vector with zeroes everywhere except for a one in the i^{th} component), and let y be e^j. Then:

$$\alpha_{ij}=x_i\alpha_{ij}y_j=x\cdot(Ay)=(Ax)\cdot y=y_j\alpha_{ji}x_i=\alpha_{ji}$$

Thus $\alpha_{ij}=\alpha_{ji}$ for all $i,j\in\{1,...,n\}$, so A is symmetric. □

A.1.2b) A function $\langle\cdot,\cdot\rangle$ from $\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}$ is an inner product on $\mathbb{R}^n$ if and only if there exists a matrix $A=(\alpha_{ij})$ that is positive definite (meaning $x\cdot(Ax)>0$ for all $x\in\mathbb{R}^n, x\neq 0$) and symmetric such that $\langle x,y\rangle=x\cdot(Ay)$ for all $x,y\in\mathbb{R}^n$.

Proof.

$\Rightarrow$ Let $\langle\cdot,\cdot\rangle$ from $\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}$ be an inner product on $\mathbb{R}^n$, and define $A=(\alpha_{ij})$ by $\alpha_{ij}=\langle e^i,e^j\rangle$. Then by the properties of the inner product:

$$x\cdot(Ay)=\Sigma_{i=1}^n x_i(Ay)_i=\Sigma_{i=1}^n x_i\Sigma_{j=1}^n\alpha_{ij}y_j=\Sigma_{i=1}^n x_i\Sigma_{j=1}^n\langle e^i,e^j\rangle y_j$$
$$=\Sigma_{i=1}^n x_i\Sigma_{j=1}^n\langle e^i,y_je^j\rangle=\Sigma_{i=1}^n x_i\langle e^i,y\rangle=\Sigma_{i=1}^n\langle x_ie_i,y\rangle=\langle x,y\rangle$$

And then, trivially: $x\cdot(Ax)=\langle x,x\rangle>0$ for all $x\in\mathbb{R}^n$ with $x\neq 0$, so A is positive definite, and $\alpha_{ij}=\langle e_i,e_j\rangle=\langle e_j,e_i\rangle=\alpha_{ji}$ so A is symmetric.

$\Leftarrow$ Let $A=(\alpha_{ij})$ be a positive definite symmetric $n\times n$ matrix such that $\langle x,y\rangle=x\cdot(Ay)$ for all $x,y\in\mathbb{R}^n$. Let $x,y\in\mathbb{R}^n$. Then:

$\langle x,y\rangle=x\cdot(Ay)=(Ax)\cdot y=y\cdot(Ax)=\langle y,x\rangle$ by part a,

$\langle x+y,z\rangle=(x+y)\cdot(Az)=(x\cdot(Az))+(y\cdot(Az))=\langle x,z\rangle+\langle y,z\rangle$ and

$\langle\alpha x,y\rangle=(\alpha x)\cdot(Ay)=\alpha(x\cdot(Ay))=\alpha\langle x,y\rangle$ by bylinearity of dot product;

$\langle x,x\rangle=x\cdot(Ax)>0$ because A is positive definite.

So $\langle\cdot,\cdot\rangle$ satisfies the definition of an inner product. □

A.1.3) Let A be a positive definite symmetric $n \times n$ matrix and $\cdot$ denote the inner product on $\mathbb{R}^n$. Then $|x \cdot (Ay)|^2 \leq [x \cdot Ax][y \cdot Ay]$ for all $x, y \in \mathbb{R}^n$ with equality holding if and only if x and y are linearly dependent.

Proof. Define $\langle x, y \rangle = x \cdot (Ay)$. Then by A.1.2b, $\langle \cdot, \cdot \rangle$ is an inner product and this theorem is an immediate consequence of the Cauchy-Schwartz inequality:

$$|x \cdot (Ay)|^2 = |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle = [x \cdot Ax][y \cdot Ay]$$

with equality if and only if x and y are linearly dependent. □

A.1.4) Let $\ell^2 = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}}; \|x\|_2 < \infty\}$, where $\|x\|_2^2 = \sum_{n=1}^{\infty} |x_n|^2$.

a. For each $x = (x_n)$ and $y = (y_n)$ in ℓ^2, the series

$$\sum_{k=1}^{\infty} x_k \bar{y}_k =: \langle x, y \rangle$$

converges in $\mathbb{C}$ (with absolute value) and defines an inner product on ℓ^2.

Proof. An inner product must live in a vector space, so I will first define this vector space.

Let $x = (x_n)$ and $y = (y_n)$ be sequences in ℓ^2 and $\alpha, \beta \in \mathbb{C}$.

Define addition of sequences by $(x+y)_n = x_n + y_n$ and scalar multiplication by $(\alpha x)_n = \alpha x_n$. The vector space properties follow immediately from the vector space properties of $\mathbb{C}$, because each index is independent. And these operations are closed in ℓ^2 because

$$\|\alpha x + \beta y\|_2^2 = \lim_{u \to \infty} \sum_{k=1}^{u} |\alpha x_k + \beta y_k|^2 \leq \lim_{u \to \infty} \sum_{k=1}^{u} (|\alpha x_k| + |\beta y_k|)^2 = \lim_{u \to \infty} \sum_{k=1}^{u} (\alpha |x_k| + \beta |y_k|)^2$$

$$= \lim_{u \to \infty} \sum_{k=1}^{u} (\alpha^2 |x_k|^2 + 2\alpha\beta |x_k||y_k| + \beta^2 |y_k|^2) \leq \lim_{u \to \infty} \sum_{k=1}^{u} (\alpha^2 |x_k|^2 + \beta^2 |y_k|^2)$$

$$= \lim_{u \to \infty} \left(\sum_{k=1}^{u} \alpha^2 |x_k|^2 + \sum_{k=1}^{u} \beta^2 |y_k|^2 \right) = \lim_{u \to \infty} \sum_{k=1}^{u} \alpha^2 |x_k|^2 + \lim_{u \to \infty} \sum_{k=1}^{u} \beta^2 |y_k|^2$$

$$= \alpha^2 \lim_{u \to \infty} \sum_{k=1}^{u} |x_k|^2 + \beta^2 \lim_{u \to \infty} \sum_{k=1}^{u} |y_k|^2 = \alpha^2 \|x\|_2^2 + \beta^2 \|y\|_2^2 < \infty.$$

Taking square roots, we have $\|\alpha x + \beta y\|_2 < \infty$, so $\alpha x + \beta y \in \ell^2$ and ℓ^2 is a vector space.

Then for all $k \in \mathbb{N}$,

$$0 \leq (|x_k| - |y_k|)^2 = |x_k|^2 - 2|x_k||y_k| + |y_k|^2 \text{ and } |x_k\bar{y}_k| = |x_k||y_k|,$$

and therefore $|x_k\bar{y}_k| \leq (|x_k|^2 + |y_k|^2)/2$, and

$$\lim_{u\to\infty} \sum_{k=1}^{u} |x_k\bar{y}_k| \leq \lim_{u\to\infty} \sum_{k=1}^{u} \frac{|x_k|^2 + |y_k|^2}{2} = \frac{1}{2}\left(\sum_{k=1}^{\infty} |x_k|^2\right)\left(\sum_{k=1}^{\infty} |y_k|^2\right) = \frac{\|x\|_2^2\,\|y\|_2^2}{2}$$

so $\sum_{k=1}^{\infty} |x_k\bar{y}_k|$ converges because it is a nonnegative series with bounded sum, and therefore $\sum_{k=1}^{\infty} x_k\bar{y}_k$ converges in $\mathbb{C}$ because $\mathbb{C}$ is a Banach space.

And now we must show that $\langle x, y\rangle$ satisfies the properties of an inner product:

$$\langle x, y\rangle = \lim_{u\to\infty} \sum_{k=1}^{u} x_k\bar{y}_k = \lim_{u\to\infty} \sum_{k=1}^{u} \overline{y_k\bar{x}_k} = \lim_{u\to\infty} \overline{\sum_{k=1}^{u} y_k\bar{x}_k} = \overline{\lim_{u\to\infty} \sum_{k=1}^{u} y_k\bar{x}_k} = \overline{\langle y, x\rangle}$$

$$\langle \alpha x, y\rangle = \lim_{u\to\infty} \sum_{k=1}^{u} (\alpha x_k)\bar{y}_k = \alpha \lim_{u\to\infty} \sum_{k=1}^{u} x_k\bar{y}_k = \alpha\langle x, y\rangle$$

$$\langle x + y, z\rangle = \lim_{u\to\infty} \sum_{k=1}^{u} (x_k + y_k)\bar{z}_k = \lim_{u\to\infty} \sum_{k=1}^{u} x_k\bar{z}_k + \lim_{u\to\infty} \sum_{k=1}^{u} y_k\bar{z}_k = \langle x, z\rangle + \langle y, z\rangle$$

$$\langle x, x\rangle = \lim_{u\to\infty} \sum_{k=1}^{u} x_k\bar{x}_k = \lim_{u\to\infty} \sum_{k=1}^{u} |x_k|^2 > 0 \text{ if } x \text{ is not the constant zero sequence.}$$

So $\langle x, y\rangle := \sum_{k=1}^{\infty} x_k\bar{y}_k$ defines an inner product on ℓ^2. ☐

b. ℓ^2 with this inner product is a Hilbert space.

Proof. Let (s^n) be a Cauchy sequence in ℓ^2, using superscripts to index the outer sequence and subscripts to index each inner sequence (i.e. s^1 is the first sequence in (s^n) and s_2^1 is the second element of the first sequence in (s^n)).
Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n, m > N; n, m \in \mathbb{N}$:

$$\|s^n - s^m\| = \left(\sum_{k=1}^{\infty} |s_k^n - s_k^m|^2\right)^{\frac{1}{2}} < \frac{\epsilon}{2} \quad \text{and therefore} \quad \sum_{k=1}^{\infty} |s_k^n - s_k^m|^2 < \frac{\epsilon^2}{4}$$

Using the fact that each term of a nonnegative series is less than or equal to the sum, we have $|s_k^n - s_k^m|^2 < \epsilon^2/4$ and $|s_k^n - s_k^m| < \epsilon/2 < \epsilon$. Thus s_k^n is a Cauchy sequence in $\mathbb{C}$, a complete metric space, and therefore converges to a limit in $\mathbb{C}$.
Using this, define a new sequence $s = (s_k)$ by:

$$s_k := \lim_{n\to\infty} s_k^n$$

Because (s^n) is Cauchy, there exists $B \in \mathbb{R}$ such that $\|s^n\|_2 < B$ for all $n \in \mathbb{N}$. Using this and the fact that the partial sums of a nonnegative series are increasing, we have

$$\sum_{k=1}^{u} |s_k^n|^2 \leq \sum_{k=1}^{\infty} |s_k^n|^2 = \|s^n\|_2^2 < B^2 < \infty$$

for all $n, u \in \mathbb{N}$. Now using the fact that the limit of a convergent bounded sequence in $\mathbb{R}$ must be less than or equal to the upper bound, we take the limit of this as $n \to \infty$ and find

$$B^2 \geq \lim_{n\to\infty} \sum_{k=1}^{u} |s_k^n|^2 = \sum_{k=1}^{u} |\lim_{n\to\infty} s_k^n|^2 = \sum_{k=1}^{u} |s_k|^2,$$

and then take the limit of that as $u \to \infty$, using the fact that a bounded increasing sequence must converge, which gives

$$B^2 \geq \lim_{u\to\infty} \sum_{k=1}^{u} |s_k|^2 = \|s\|_2^2\,,$$

so $s \in \ell^2$.
Similarly, because $|s_k^n - s_k^m|^2 \geq 0$,

$$\sum_{k=1}^{u} |s_k^n - s_k^m|^2 \leq \sum_{k=1}^{\infty} |s_k^n - s_k^m|^2 = \|s^n - s^m\|_2^2 < \frac{\epsilon^2}{4}$$

for all $n, m > N; n, m \in \mathbb{N}$ and all $u \in \mathbb{N}$, and therefore, now taking the limit as $m \to \infty$,

$$\epsilon^2 > \frac{\epsilon^2}{4} \geq \lim_{m\to\infty} \sum_{k=1}^{u} |s_k^n - s_k^m|^2 = \sum_{k=1}^{u} \left| \lim_{m\to\infty} s_k^n - \lim_{m\to\infty} s_k^m \right|^2 = \sum_{k=1}^{u} |s_k^n - s_k|^2$$

for all $n > N; n \in \mathbb{N}$, and then, taking the limit as $u \to \infty$,

$$\epsilon^2 > \frac{\epsilon^2}{4} \geq \lim_{u\to\infty} \sum_{k=1}^{u} |s_k^n - s_k|^2 = \|s^n - s\|_2^2$$

we find that $\|s^n - s\|_2 < \epsilon$ for all $n > N; n \in \mathbb{N}$. Therefore $s^n \to s \in \ell^2$ as $n \to \infty$ and because (s_n) was arbitrary, this means ℓ^2 is complete. We've already shown that ℓ^2 is an inner product space, so ℓ^2 is a Hilbert space. □

Lemma 1) Let X be an inner product space over $\mathbb{K}$ and (s_n) be a Cauchy sequence in X. Then there exists a real number $U > 0$ such that $\|s_n\| < U$ for all $n \in \mathbb{N}$.

Proof. Because (s_n) is a Cauchy sequence in X, there exists $N \in \mathbb{N}$ such that for all $n, m > N; n, m \in \mathbb{N}$ $\|s_m - s_n\| < 1$.
Let $M = \max\{\|s_1\|, \|s_2\|, ... \|s_N\|, \|s_{N+1}\|\}$, which must exist because it is the maximum of a finite set of real numbers. We know $M \geq 0$ because each norm is nonnegative.
If $k \leq N+1$, then $\|s_k\| \leq M < M+1$,
If $k > N+1$:

$$\|s_k\| = \|s_{N+1} + (s_k - s_{N+1})\| \leq \|s_{N+1}\| + \|s_k - s_{N+1}\| < M + 1$$

Let $U = M + 1$. Then for all $k \in \mathbb{N}$, $\|s_k\| < U$, as required.

□

A.1.5) Let X be an inner product space over $\mathbb{K}$ and $(x_n), (y_n)$ be Cauchy sequences in X. Then the sequence $(\langle x_n, y_n \rangle)$ converges in $\mathbb{K}$.

Proof. Let $\epsilon > 0$.
Because (x_n) and (y_n) are Cauchy sequences in X, by lemma 1 there exist $U, V > 0$ such that $\|x_k\| < U$ and $\|y_k\| < V$ for all $k \in \mathbb{N}$.
Let $M = \max(U, V)$, and let $\delta = \min(1, \epsilon/(2M+1)) > 0$. Again because (x_n) and (y_n) are Cauchy sequences in X, there exists $N \in \mathbb{N}$ such that for all $n, m > N; n, m \in \mathbb{N}$, $\|x_m - x_n\| < \delta$ and $\|y_m - y_n\| < \delta$. Then by the properties of inner products and the Cauchy-Schwartz inequality, for all $n, m > N; n, m \in \mathbb{N}$:

$$|\langle x_m, y_m \rangle - \langle x_n, y_n \rangle| = |\langle x_n + (x_m - x_n), y_n + (y_m - y_n) \rangle - \langle x_n, y_n \rangle|$$

$$= |\langle x_n, y_n \rangle + \langle x_m - x_n, y_n \rangle + \langle x_n, y_m - y_n \rangle + \langle x_m - x_n, y_m - y_n \rangle - \langle x_n, y_n \rangle|$$

$$= |\langle x_m - x_n, y_n \rangle + \langle x_n, y_m - y_n \rangle + \langle x_m - x_n, y_m - y_n \rangle|$$

$$\leq |\langle x_m - x_n, y_n \rangle| + |\langle x_n, y_m - y_n \rangle| + |\langle x_m - x_n, y_m - y_n \rangle|$$

$$\leq \|x_m - x_n\| \|y_n\| + \|x_n\| \|y_m - y_n\| + \|x_m - x_n\| \|y_m - y_n\|$$

$$< \delta M + M\delta + \delta^2 = \delta(2M + \delta) \leq \delta(2M + 1) \leq \frac{\epsilon}{2M+1}(2M+1) = \epsilon.$$

Thus $(\langle x_n, y_n \rangle)$ is a Cauchy sequence in $\mathbb{K}$, a complete metric space, so $(\langle x_n, y_n \rangle)$ converges in $\mathbb{K}$. □

A.1.6) Let X be an inner product space over $\mathbb{K}$ with norm $\|\cdot\|$ induced by the inner product, and let x, y be points in X, $\alpha \in \mathbb{K}$, and $(x_n), (y_n)$ be sequences in X and (α_n) a sequence in $\mathbb{K}$. If $x_n \to x, y_n \to y$, and $\alpha_n \to \alpha$ as $n \to \infty$, then $\langle \alpha_n x_n, y_n \rangle \to \langle \alpha x, y \rangle$ as $n \to \infty$.

Proof. Let $\epsilon > 0$, and let $x_n \to x, y_n \to y$, and $\alpha_n \to \alpha$ as $n \to \infty$.
Let $A = \max(|\alpha|, \|x\|)$, and let $\gamma = \min(1, \epsilon/(2A+1)) > 0$. Because $\alpha_n \to \alpha$ and $x_n \to x$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n > N; n \in \mathbb{N}$: $|\alpha_n - \alpha| < \gamma$ and $\|x_n - x\| < \gamma$, and therefore:

$$\begin{aligned}
\|\alpha_n x_n - \alpha x\| &= \|[\alpha + (\alpha_n - \alpha)][x + (x_n - x)] - \alpha x\|, \\
&= \|\alpha x + \alpha(x_n - x) + (\alpha_n - \alpha)x + (\alpha_n - \alpha)(x_n - x) - \alpha x\| \\
&= \|\alpha(x_n - x) + (\alpha_n - \alpha)x + (\alpha_n - \alpha)(x_n - x)\| \\
&\leq \|\alpha(x_n - x)\| + \|(\alpha_n - \alpha)x\| + \|(\alpha_n - \alpha)(x_n - x)\| \\
&= |\alpha| \|(x_n - x)\| + |\alpha_n - \alpha| \|x\| + |\alpha_n - \alpha| \|(x_n - x)\| \\
&< \gamma A + A\gamma + \gamma^2 = \gamma(2A + \gamma) \leq \gamma(2A + 1) \leq \frac{\epsilon}{2A+1}(2A+1) = \epsilon,
\end{aligned}$$

which shows that $\alpha_n x_n \to \alpha x$ as $n \to \infty$.

Let $M = \max(\|\alpha x\|, \|y\|)$, and let $\delta = \min(1, \epsilon/(2M+1)) > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n > N; n \in \mathbb{N}$: $\|\alpha_n x_n - \alpha x\| < \delta$ and $\|y_n - y\| < \delta$, and therefore:

$$\begin{aligned}
|\langle \alpha_n x_n, y_n \rangle - \langle \alpha x, y \rangle| &= |\langle \alpha x + (\alpha_n x_n - \alpha x), y + (y_n - y) \rangle - \langle \alpha x, y \rangle| \\
&= |\langle \alpha x, y \rangle + \langle \alpha_n x_n - \alpha x, y \rangle + \langle \alpha x, y_n - y \rangle + \langle \alpha_n x_n - \alpha x, y_n - y \rangle - \langle \alpha x, y \rangle| \\
&= |\langle \alpha_n x_n - \alpha x, y \rangle + \langle \alpha x, y_n - y \rangle + \langle \alpha_n x_n - \alpha x, y_n - y \rangle| \\
&\leq |\langle \alpha_n x_n - \alpha x, y \rangle| + |\langle \alpha x, y_n - y \rangle| + |\langle \alpha_n x_n - \alpha x, y_n - y \rangle| \\
&\leq \|\alpha_n x_n - \alpha x\| \|y\| + \|\alpha x\| \|y_n - y\| + \|\alpha_n x_n - \alpha x\| \|y_n - y\| \\
&< \delta M + M\delta + \delta^2 = \delta(2M + \delta) \leq \delta(2M + 1) \leq \frac{\epsilon}{2M+1}(2M+1) = \epsilon,
\end{aligned}$$

and thus $\langle \alpha_n x_n, y_n \rangle \to \langle \alpha x, y \rangle$ as $n \to \infty$. $\square$

A.1.7) Let X be an inner product space, $x \in X$, and (x_n) a sequence in X. Then $x_n \to x$ as $n \to \infty$ if and only if $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$.

Proof.

$\Rightarrow$ Let $x_n \to x$ as $n \to \infty$. Then $\|x_n - x\| \to 0$ as $n \to \infty$, and by the reverse triangle inequality,

$$| \, \|x_n\| - \|x\| \, | \leq \|x_n - x\| \to 0 \text{ as } n \to \infty,$$

so $\|x_n\| \to \|x\|$ as $n \to \infty$. And by the Cauchy-Schwartz inequality,

$$\|\langle x_n, x \rangle - \langle x, x \rangle\| = \|\langle x_n - x, x \rangle\| \leq \|x_n - x\| \, \|x\| \to 0 \text{ as } n \to \infty,$$

so $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$.

$\Leftarrow$ Let $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$. Then $\langle x_n, x_n \rangle = \|x_n\|^2 \to \|x\|^2 = \langle x, x \rangle$ and $\langle x, x_n \rangle = \overline{\langle x_n, x \rangle} \to \overline{\langle x, x \rangle} = \langle x, x \rangle$ as $n \to \infty$, and

$$\|x_n - x\|^2 = \langle x_n - x, x_n - x \rangle$$

$$= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle$$

$$\to \langle x, x \rangle - \langle x, x \rangle - \langle x, x \rangle + \langle x, x \rangle = 0 \text{ as } n \to \infty,$$

so $x_n \to x$ as $n \to \infty$. $\square$

A.1.8) Let X be an inner product space. Let $y \in X$ be fixed but arbitrary. Define $f, g : X \to \mathbb{C}$ by

$$f(x) = \langle x, y \rangle \quad g(x) = \langle y, x \rangle, \quad x \in X$$

Then f and g are Lipschitz continuous with Lipschitz constant $\|y\|$.

Proof. By the Cauchy-Schwartz inequality, for all $u, v \in X$:

$$\|f(u) - f(v)\| = \|\langle u, y \rangle - \langle v, y \rangle\| = \|\langle u - v, y \rangle\| \leq \|y\| \, \|u - v\| \, ,$$
$$\|g(u) - g(v)\| = \|\langle y, u \rangle - \langle y, v \rangle\| = \|\langle y, u - v \rangle\| \leq \|y\| \, \|u - v\| \, .$$

Thus f and g are Lipschitz continuous with Lipschitz constant $\|y\|$. □

A.1.9) Let M be a complete linear subspace of the inner product space X. Then each vector $u \in X$ has a unique representation $u = v + w$ such that $v \in M$ and $\langle w, z \rangle = 0$ for all $z \in M$.

Proof. Let $u \in X$. Since every linear subspace is a convex set, M is a complete convex set in X, so by proposition A.9 there exists a unique $v \in M$ such that $d(u, M) = \|u - v\|$.
Let $w = u - v$. Clearly $u = v + w$.
To show that $\langle w, z \rangle = 0$ for all $z \in M$, let $z \in M$. If $z = 0$ then trivially $\langle w, z \rangle = 0$.
If $z \neq 0$, suppose $\langle w, z \rangle \neq 0$ and let $\nu = (v + \frac{\langle w, z \rangle}{\|z\|^2} z)$. Then $\nu \in M$ because $\frac{\langle w, z \rangle}{\|z\|^2} \in \mathbb{K}$, $\nu \neq v$ because $\frac{\langle w, z \rangle}{\|z\|^2} \neq 0$, and

$$d(u, \nu)^2 = \|u - \nu\|^2 = \left\| u - (v + \frac{\langle w, z \rangle}{\|z\|^2} z) \right\|^2$$
$$= \left\| w - \frac{\langle w, z \rangle}{\|z\|^2} z \right\|^2 = \langle w - \frac{\langle w, z \rangle}{\|z\|^2} z, w - \frac{\langle w, z \rangle}{\|z\|^2} z \rangle$$
$$= \langle w, w \rangle - \frac{\langle w, z \rangle}{\|z\|^2} \langle z, w \rangle - \frac{\overline{\langle w, z \rangle}}{\|z\|^2} \langle w, z \rangle + \frac{\langle w, z \rangle \overline{\langle w, z \rangle}}{\|z\|^4} \langle z, z \rangle$$
$$= \|w\|^2 - \frac{|\langle w, z \rangle|^2}{\|z\|^2} < \|w\|^2 = \|u - v\|^2 = d(u, M)^2.$$

Taking square roots, we find

$$d(u, \nu) < d(u, M)$$

which is a contradiction because $\nu \in M$, so we've disproven the assumption that $\langle w, z \rangle \neq 0$, and therefore $\langle w, z \rangle = 0$. Because z was arbitrary, $\langle w, z \rangle = 0$ for all $z \in M$.
To show uniqueness, let $u = \tilde{v} + \tilde{w}$ be another representation such that $\tilde{v} \in M$ and $\langle \tilde{w}, z \rangle = 0$ for all $z \in M$. Then $\tilde{v} + \tilde{w} = u = v + w$, so $\tilde{w} - w = v - \tilde{v} \in M$ so $\tilde{w} - w \in M$, and

$$\langle \tilde{w} - w, \tilde{w} - w \rangle = \langle \tilde{w}, \tilde{w} - w \rangle - \langle w, \tilde{w} - w \rangle = 0$$

and therefore $\tilde{w} - w = 0$ and $\tilde{w} = w$ by the contrapositive of property (iv) of inner products. Then $\tilde{v} = u - \tilde{w} = u - w = v$, and the representation $u = v + w$ is unique. □