1. For each $f \in C[0,1]$ define $\Phi f \in C[0,1]$ by $\Phi f(x) = \int_0^x f(t) \, dt$. Prove that every sequence in the set $\mathcal{F} := \{ \Phi f : \| f \|_u \leq 1 \}$ has a convergent subsequence. Note: Throughout this exam, $C[0,1]$ and all other function spaces consist of complex-valued functions.

2. Use the Stone-Weierstrass Theorem to help prove the well-known result that $C[0,1]$ is separable. Include all details.

3. Use the Hahn-Banach Theorem and the Riesz Representation Theorem for $L^p$ duality to prove that if $\mu$ is $\sigma$-finite then for every $f \in L^{7/4}(X, \mathcal{M}, \mu)$ there is a $g \in L^{7/3}(X, \mathcal{M}, \mu)$ such that $(\int_X |g|^{7/3} \, d\mu)^{3/7} = 1$ and

$$\int_X fg \, d\mu = \left( \int_X |f|^{7/4} \, d\mu \right)^{4/7}.$$

4. Let $f : [0,1] \to [0,1]$ be the Cantor function, and define $g : [0,1] \to \mathbb{R}$ by $g(x) = f(x) + x$. It is a standard fact that $g$ is a homeomorphism of $[0,1]$ onto $[0,2]$. Give the proofs of the following well-known facts:

(a) Letting $C$ denote the Cantor set, the Lebesgue measure of $g(C)$ is 1.

(b) There exists a Lebesgue measurable set $A \subset [0,1]$ such that $g(A)$ is not Lebesgue measurable.

5. (a) Give an elementary argument showing that if $t < 1$ then $x^t/(1+x^2)$ is Lebesgue integrable on $(0, \infty)$.

(b) Find $\lim_{n \to \infty} \int_0^\infty \frac{(\sqrt{x})^{1/n}}{1 + x^2} \, dx$, with proof.

6. Consider the following two Borel measures on $[0,1]$: Lebesgue measure $m$ and counting measure $\mu$.

(a) Prove that $m$ is absolutely continuous with respect to $\mu$, but $m$ has no Radon-Nikodym derivative with respect to $\mu$.

(b) Prove that $\mu$ has no Lebesgue decomposition with respect to $m$, that is, there do not exist measures $\lambda$ and $\rho$ such that $\mu = \lambda + \rho$, $\lambda \perp m$, and $\rho \ll m$.

Hint: in both parts (a) and (b) consider singleton sets $\{x\}$.