Analysis Qualifying Exam (Part 1)

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School of Mathematical and Statistical Sciences,
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Instructions

Write your solutions on the blank pages provided. Write on only one side of each page, and leave reasonable margins. (Papers will be photocopied before grading.) Put your name on each page that you submit, and number the pages sequentially.

DO NOT STAPLE OR FOLD THE PAGES

Turn in this cover sheet and the exam problem page with your solutions.

No notes, books, calculators, or outside assistance is allowed.

Be sure to read and follow the instructions for each problem. You may use any part of a problem (solved or not) in the solution of any other problem, and in any later part of the same problem.

Unless otherwise indicated, you may use any results from the course lectures and from any homework you turned in. When you use such results, be sure to make some acknowledgement: for example, “since $\mathbb{Q}$ is dense in $\mathbb{R}, . . .”$ or “by the Algebra of Limits . . .”. Also, if you apply such a result, carefully show that all assumptions are satisfied.

NAME:__________________________________________________________

EMAIL:________________________________________________________
1. Let $X$ be a metric space and $d$ the metric. Show from scratch: If $S$ is a compact subset of $X$, then $S$ is a complete subset of $X$.

2. Let $X$ be a normed vector space and $f : X \to \mathbb{R}$. Assume that $f$ is differentiable at any $x \neq 0$.
   (a) Show that, for fixed $x \in X, x \neq 0$, the real function $g$ given by $g(t) = f(tx)$ is differentiable and find $g'(t)$.

   (b) Assume that there exists a linear bounded map $L$ from $X$ to $\mathbb{R}$ such that $Ly = \lim_{x \to 0} Df(x)y$ for each $y \in X$.
   Here $Df(x)$ is the Frechet derivative of $f$ at $x$.
   Show that $f$ is differentiable at 0 and $Df(0) = L$.
   (You can use the mean value theorem in $\mathbb{R}$ without proof.)

3. Show that
   \[ \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \left( n - \frac{1}{2} \right) \pi + \frac{t}{n} \right), \]
   converges uniformly in $t \in [-a, a]$ for every $a > 0$ and provides a continuously differentiable function on $\mathbb{R}$.

4. Consider the integral equation
   \[ u(x) = \int_{0}^{1} k(t) \sin(u(x + t))dt + f(x), \quad x \geq 0. \]
   Here $k : \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \to \mathbb{R}$ are given, $k$ is continuous, $\int_{0}^{1} |k(t)|dt < 1$, and $f$ is continuous and bounded, $\mathbb{R}_+ = [0, \infty)$.
   (a) Show that there exists a unique solution $u \in BC(\mathbb{R}_+)$. Here $BC(\mathbb{R}_+)$ is the Banach space of bounded continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ with the supremum norm $\|u\|_\infty = \sup_{x \geq 0} |u(x)|$. (You can use without proof that it is a Banach space, indeed.)

   (b) Let $v \in BC(\mathbb{R}_+)$ be a solution to
   \[ v(x) = \int_{0}^{1} k(t) \sin(v(x + t))dt + g(x), \quad x \geq 0, \]
   with a given $g \in BC(\mathbb{R}_+)$. Show that there exists some $M > 0$ (that does not depend on $f$ and $g$) such that
   \[ \|u(x) - v(x)\|_\infty \leq M\|f - g\|_\infty. \]