INSTRUCTIONS

Solve SEVEN of the following ten problems. Do NOT submit more than seven problems. Write the
numbers of the problems you submit on the first page of your exam.

Write your solutions on the blank paper provided. Write on only one side of each sheet, and leave
reasonable margins (the papers will be photocopied before grading).

Write your name in the upper right-hand corner of every page that you submit.

IMPORTANT

(*) Start each problem on a new page.

(*) Do not staple or fold corners of the pages you submit.

Give complete proofs, written in complete sentences, that show all of your reasoning. Be sure to verify
all hypotheses of any theorem that you use, and also give the name of the theorem if it has one. You may
ignore any hints, but you must follow all explicit instructions.

If you are stuck on a problem, go on to the others before coming back to the difficult one.

No notes, books, calculators, or electronic equipment of any kind may be used during the exam. Please
silence all such equipment, especially cell phones, and put them away during the exam.
1. Let \((X,d)\) be a metric space. We write \(B_r(x) := \{y \in X : d(y,x) \leq r\}\). Suppose that
   (i) for each \(x \in X\), there exists \(r > 0\) such that \(B_r(x)\) is compact;
   (ii) \(X\) contains a countable dense subset.

   Prove that \(X\) is \(\sigma\)-compact, i.e. that \(X\) equals the union of countably many compact subsets. (Hint: for \(x \in X\) consider \(R(x) := \sup\{r > 0 : B_r(x)\) is compact\}.)

2. Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. Assume that \(Y\) is connected. Let \(f : X \to Y\) be a continuous surjective function. Suppose that
   (i) for each \(y \in Y\), \(f^{-1}(y)\) is compact and connected;
   (ii) for each \(y_0 \in Y\), \(\lim_{y \to y_0} \text{dist}(f^{-1}(y), f^{-1}(y_0)) = 0\).

   Prove that \(X\) is connected. (Recall that for subsets \(A, B \subseteq X\), the distance between \(A\) and \(B\) is defined to be \(\text{dist}(A,B) := \inf\{d_X(a,b) : a \in A, b \in B\}\).)

3. Let \(f, f_n : [0, \infty) \to \mathbb{C}\) be functions, \(n = 1, 2, 3, \ldots\). Suppose that \(f_n \to f\) uniformly on \([0,\infty)\). Suppose also that for each \(n\), \(\lim_{x \to \infty} f_n(x)\) exists. Prove that \(\lim_{x \to \infty} f(x)\) exists.

4. Let \((X,M,\mu)\) be a measure space.
   (a) Let \(E_1, E_2, \ldots \in M\) be such that \(\mu(E_n) < \infty\) for all \(n\). Let \(F \in M\) and suppose that \(\chi_{E_n} \to \chi_F\) in \(\mu\)-measure. Prove that \(\mu(F) < \infty\).
   (b) Let \(f_n \in L^1(X,M,\mu)\), let \(f : X \to \mathbb{C}\) be a measurable function, and suppose that \(f_n \to f\) in \(\mu\)-measure. Does it follow that \(f \in L^1(X,M,\mu)\)? Prove your answer.

5. Let \(f : \mathbb{R} \to \mathbb{C}\) be a measurable function. Let \(g : \mathbb{R}^2 \to \mathbb{R}\) be given by \(g(x,y) = f(x) - f(y)\). Suppose that \(g\) is integrable over \(\mathbb{R}^2\). Prove that there is \(z_0 \in \mathbb{C}\) such that \(f(x) = z_0\) for almost all \(x \in \mathbb{R}\).

6. In this problem we use the following notation: for \(x \in \mathbb{R}\) let \((x)_+ = x\), if \(x \geq 0\), and \((x)_+ = 0\) if \(x < 0\).

   Let \(F : [a,b] \to \mathbb{R}\). Define \(P_F : [a,b] \to [0,\infty)\) by
   \[P_F(x) = \sup \left\{ \sum_{i=1}^{n} (F(t_i) - F(t_{i-1}))_+ : a = t_0 < t_1 < \cdots < t_n = x \right\}.\]
   (a) Prove that \(P_F\) is an increasing function (i.e., that if \(x < y\) then \(P_F(x) \leq P_F(y)\)).
   (b) Suppose that \(P_F(x) < \infty\) for all \(x \in \mathbb{R}\). Prove that \(F\) is of bounded variation on \([a,b]\). (Hint: consider \(P_F - F\).)

7. Let \(E \subseteq \mathbb{R}^n\) be a Lebesgue measurable set. A point \(x \in \mathbb{R}^n\) is called a density point of \(E\) if
   \[\lim_{r \to 0+} \frac{m(B_r(x) \cap E)}{m(B_r(x))} = 1.\]

   Prove that almost every point of \(E\) is a density point of \(E\).

8. Let \((X,M,\mu)\) be a finite measure space. Let \(\mu_1, \mu_2, \ldots\) be complex measures on \((X,M)\). Suppose that
   (i) for each \(E \in M\), \(\lim_{n \to \infty} \mu_n(E)\) exists;
   (ii) for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that for each \(n \in \mathbb{N}\) and for each \(E \in M\), if \(\mu(E) < \delta\) then \(|\mu_n(E)| < \varepsilon\).

   Define \(\nu : M \to \mathbb{C}\) by \(\nu(E) = \lim_{n \to \infty} \mu_n(E)\). Prove that \(\nu\) is a complex measure on \((X,M)\).

9. Let \((X,M,\mu)\) be a finite measure space, let \(1 < p < \infty\), and let \(f \in L^p(X,M,\mu)\). Prove that \(\lim_{r \to 0^+} \|f_r\| = \|f\|_p\).

10. Let \((X,M,\mu)\) be a finite measure space. Let \(f : X \to \mathbb{C}\) be a measurable function such that \(f > 0\) on \(X\). Let \(0 < \alpha < \mu(X)\). Prove that \(\inf \{ \int_E f \, d\mu : \mu(E) \geq \alpha \} > 0\).