Fields:

1. Let $p$ be prime, and set $f(x) = x^p - x - c \in \mathbb{F}_p[x], c \neq 0$.
   
   (a) (10 pts) Determine the splitting field of $f(x)$.
   
   (b) (10 pts) Show that the Galois group of $f(x)$ is cyclic of order $p$.

2. (a) (10 pts) Let $K_1, K_2$ be finite extensions of a field $F$. Prove that

   \[ [K_1K_2 : F] \leq [K_1 : F][K_2 : F]. \]

   (b) (10 pts) Suppose that $[K_1 : F] = m$ and $[K_2 : F] = n$ where $(m, n) = 1$. Prove that $[K_1K_2 : F] = [K_1 : F][K_2 : F] = mn$.

3. Let $E/F$ be a field extension.

   (a) (15 pts) Sketch a proof that if $\alpha, \beta \in E$ are algebraic over $F$, then $\alpha \pm \beta, \alpha\beta, \alpha/\beta$ ($\beta \neq 0$) are algebraic over $F$.

   (b) (10 pts) Give an example of algebraic numbers $\alpha, \beta$ such that

   \[ |\mathbb{Q}(\alpha) : \mathbb{Q}| > |\mathbb{Q}(\beta) : \mathbb{Q}| > |\mathbb{Q}(\alpha\beta) : \mathbb{Q}|. \]

4. (30 pts) Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of degree 4, having roots $a_i, i = 1, \ldots, 4$, in an extension field of $\mathbb{Q}$. Define (in terms of the $a_i$) the discriminant $D$ of $f(x)$ and the resolvent cubic $g(x)$ of $f(x)$. Describe how a knowledge of $D$ and of $g(x)$ allows computation of the Galois group of $f(x)$. 
5. (a) (10 pts) If \( G \) is a finite Abelian group, then \( G \) is naturally a \( \mathbb{Z} \)-module. Can this action be extended to make \( G \) into a \( \mathbb{Q} \)-module?

(b) (10 pts) Give an explicit example of a map from one \( R \)-module to another which is a group homomorphism but not an \( R \)-module homomorphism.

(c) (10 pts) Exhibit all \( \mathbb{Z} \)-module homomorphisms from \( \mathbb{Z}/21\mathbb{Z} \) to \( \mathbb{Z}/15\mathbb{Z} \).

6. (a) (10 pts) State the Cayley-Hamilton Theorem for an \( n \times n \) matrix \( A \) with real entries, and prove it under the assumption that \( A \) is diagonalizable.

(b) (10 pts) Let \( V \) denote the vector space of all \( n \times n \) matrices with real entries, and consider the linear transformation \( T : V \to V \) defined by \( T(A) = A^t \), where \( A^t \) denotes the transpose of \( A \). Determine the minimum polynomial and characteristic polynomial of \( T \).

7. Let \( N \) be an \( n \times n \) matrix with coefficients in the field \( F \). Suppose \( N \) is nilpotent, that is, \( N^k = 0 \) for some positive integer \( k \).

(a) (10 pts) Prove that \( N \) is similar to a block diagonal matrix whose blocks are matrices with 1’s on the first superdiagonal, and 0’s elsewhere.

(b) (10 pts) Prove that if \( N \) is an \( n \times n \) nilpotent matrix, then \( N^n = 0 \). (You should not quote the Cayley-Hamilton Theorem).

8. (a) (15 pts) Classify up to similarity all \( 3 \times 3 \) matrices \( A \) over \( \mathbb{Q} \) satisfying \( A^8 = I \). Give reasons why your classification is complete.

(b) (10 pts) Classify up to similarity all \( 3 \times 3 \) matrices \( A \) over \( \mathbb{Z}/2\mathbb{Z} \) satisfying \( A^8 = I \).