1. Let $G$ be a group with $H, K$ normal subgroups of $G$ such that $H \cap K = \{e\}$.
   (a) (5 pts) Prove $HK = KH$
   (b) (5 pts) Suppose $G = HK$. Prove that $G \simeq H \times K$.
   (c) (5 pts) Suppose $G = HK$ and $N \triangleleft G$ such that $N \cap H = \{e\} = N \cap K$. Prove that $N \leq Z(G)$.
   (d) (5 pts) Suppose $N_1, N_2, N_3$ are normal subgroups of $G$ such that $N_i \cap N_j = \{e\}$ for all $i \neq j$ and $N_iN_j = G$ for all $i \neq j$. Prove that $G$ is Abelian, and $N_i \simeq N_j$ for all $i, j$.

2. Let $G$ be a group.
   (a) (5 pts) Prove that the set of all inner automorphisms of $G$ is a subgroup of $\text{Aut}(G)$, which is normal in $\text{Aut}(G)$.
   (b) (5 pts) Prove that the group of inner automorphisms of $G$ is isomorphic to $G/Z(G)$.
   (c) (10 pts) Let $H \leq G$. Prove that $\text{Aut}(H)$ has a subgroup isomorphic to $N_G(H)/C_G(H)$.

3. (a) (5 pts) Let $H, K$ be groups and let $\phi$ be a homomorphism from $K$ into $\text{Aut}(H)$. Define the semidirect product $H \rtimes K$.
   (b) (15 pts) Determine up to isomorphism all groups of order 21.

4. (a) (12 pts) Let $G$ be a finite simple group whose order is divisible by at least two primes. Let $p | |G|$ and let $n_p$ be the number of Sylow $p$-subgroups of $G$. Prove that $|G|$ divides $n_p!$
   (b) (8 pts) Prove that there are no simple groups of order 1,000,000
5. Let $R$ be a commutative ring with one.

(a) (5 pts) Suppose $x \in R$ is nilpotent. Prove that $1 + x$ is a unit.
(b) (5 pts) Suppose $x, y \in R$ are nilpotent. Prove that $x + y$ is nilpotent.
(c) (10 pts) Suppose $R$ is commutative. Prove that $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in R[x]$ is a unit if and only if $a_0$ is a unit and $a_i$ is nilpotent for all $0 < i \leq n$.

6. Let $R$ be a commutative ring with 1, and let $J$ be be the intersection of all maximal ideals in $R$.

(a) (8 pts) Prove that $J$ is an ideal of $R$
(b) (12 pts) Prove that $x \in J$ if and only if $1 - xy$ is a unit of $R$ for all $y \in R$.

7. Let $R$ be a Unique Factorization Domain.

(a) (12 pts) Sketch a proof that the polynomial ring $R[x]$ is a Unique Factorization Domain
(b) (8 pts) Let $f(x) \in R[x]$ possess a zero of multiplicity $m$ in the field of fractions of $R$. Prove that either the zero is a unit of $R$, or at least one of the coefficients of $f(x)$ is divisible by the $m$-th power of an irreducible element of $R$.

8. True or false? Give a proof or a counterexample as appropriate.

(a) (6 pts) If $R$ is a Unique Factorization Domain, then the highest common factor $(a, b)$ of any two non-zero elements $a, b \in R$ can be written in the form $ra + sb$ for suitable $r, s \in R$.
(b) (7 pts) If $R$ is an Integral Domain, and if any two non-zero elements of $R$ have a highest common factor, then $R$ is a Unique Factorization Domain.
(c) (7 pts) If $R$ is a Unique Factorization Domain, then any two non-zero elements of $R$ have a least common multiple which is uniquely determined up to multiplication by a unit of $R$. 