Graph Theory Qualifier

August 21, 2012

Directions. Solve all six problems below. Ask questions if any of the wording is ambiguous or confusing.

1. (10 pts.) Prove that for every integer \( k \geq 2 \) there exists a graph \( G \) with \( \alpha(G), \omega(G) < k \) and \( |G| \geq 2^{(k-1)/2} \).

2. (10 pts.) Show that if \( G \) is a graph on \( n \geq 3 \) vertices with \( \delta(G) \geq \frac{n}{2} \), then \( G \) contains a Hamilton cycle.

3. (10 pts.) Let \( G = (V,E) \) be a planar graph with \( \delta(G) = 5 \).
   
   (a) Suppose \( S \subseteq V \), and \( H \) be a component of \( G - S \) with \( |H| = 3 \). Show that \( |\bigcup_{x \in V(H)} (S \cap N(x))| \geq 4 \).
   
   (b) Prove that \( G \) has a matching of size at least \( \frac{2}{5}|G| \), i.e., at most \( \frac{1}{5}|G| \) unsaturated vertices.

4. (10 pts.) Prove: Every tree \( T \) containing \( 2k \) vertices with odd degree (and maybe some vertices with even degree) decomposes into \( k \) paths.

5. (10 pts.) Suppose \( G := K_{s,t} \) is an \( S,T \)-bighraph, where \( |S| = s \). Prove: \( G \) is \( s \)-chosable iff \( t < s^2 \). [Hint: First prove that if \( L \) is an \( s \)-list assignment for \( G \) with \( L(v) \cap L(w) \neq \emptyset \) for distinct \( v, w \in S \) then \( G \) is \( L \)-colorable.]

6. (10 pts.) Suppose \( G \) is a three connected graph. Prove: For all vertices \( x, y \in V(G) \), there exists a partition \( \{V_1, V_2\} \) of \( V(G) \) such that \( G[V_1] \) is an \( x, y \)-path and \( G[V_2] \) is connected. [Hint: Contract an edge.]