CHAPTER 6: THE LAPLACE TRANSFORM

6.1: Definition of the Laplace Transform

- The Laplace transform of $f(t)$ is defined through an improper integral $\int_0^\infty f(t)e^{-st}\,dt.$ Be able to calculate the transform of basic functions using the definition.
- Remember that the Laplace transform does not exist for all functions. The functions need to be piecewise continuous and of exponential order for the Laplace transform to exist.
- Know how to compute the Laplace transform of functions using the tables. A preliminary algebraic manipulation may be necessary.

6.2: Solution of Initial Value Problems

- Know how to transform derivatives of functions:

  (1) $\mathcal{L}\{y'(t)\}=s\,Y(s)-y(0)$

  (2) $\mathcal{L}\{y''(t)\}=s^2\,Y(s)-s\,y(0)-y'(0)$

  and in general:

  (3) $\mathcal{L}\{y^{(n)}(t)\}=s^n\,Y(s)-s^{n-1}\,y(0)-s^{n-2}\,y'(0)-\ldots-y^{(n-1)}(0)$

  where $Y(s)=\mathcal{L}\{y(t)\}$
- Know how to compute inverse transform functions. You will have to use some algebraic manipulation and/or partial fractions decomposition (PFD) to put it in a form that can be found in the Table.
- Know how to solve linear differential equations using the Laplace transform. This involves three steps:
  - Apply the Laplace transform to both sides of the equation and use formulas (1),(2) and/or (3) above so that the equation contains the Laplace of *y* only (which we denote by *Y*(*s*)).
  - Substitute the initial conditions and solve for *Y*(s).
  - Compute the inverse transform of *Y*(*s*). This is the solution to the DE.
- Know how to apply the Differentiation of Transforms property: $\mathcal{L}\{t^n f(t)\}=(-1)^n F^{(n)}(s)$

6.3: Step Functions

- Know the definition of the unit step function $u_c(t)=u(t-c)$ and how to write a piecewise function in terms of the unit step functions and use the appropriate entry in the table to find the Laplace transform.
- Know how to graph functions involving the unit step function.
- Know how to apply Theorem 6.3.1:

  $\mathcal{L}\{u_c(t)f(t-c)\}=e^{-cs}\mathcal{L}\{f(t)\}$

  and, conversely, if $f(t)=\mathcal{L}^{-1}\{F(s)\}$, then $u_c(t)f(t-c)=\mathcal{L}^{-1}\{e^{-cs}F(s)\}$
- Know how to use the translation property: $\mathcal{L}\{e^{ct}f(t)\}=F(s-c)$

6.4: Differential Equations with Discontinuous Forcing Equations.

Know how to solve Differential Equations where the forcing term is given by a piecewise continuous function. In these cases the function needs to be written in terms of unit step functions $u_c(t)$ in order to evaluate the Laplace.

6.5: Impulse Functions

Know the definition of the delta function, $\delta_c(t)$, and know how to solve differential equations where the forcing terms involves delta functions.

CHAPTER 7: LINEAR SYSTEMS

Section 7.3: Linear Independence, Eigenvalues and Eigenvectors

- Know the definition of linearly independent and linearly dependent vectors.
- Know how to find eigenvalues and eigenvectors of a matrix.

Section 7.4: Basic Theory of Systems of First Order Linear Equations

- Know how to write a linear system in the form $\mathbf{x}'=P(t)\,\mathbf{x}+\mathbf{f}(t)$ with $P(t)$ the coefficient matrix and **x** and $\mathbf{f}(t)$ the appropriate vectors.
- Know how to verify (by substituting) that a given vector function is a solution to a given system.
- Given an $n\times n$ homogeneous linear system of differential equations written in matrix form as $\mathbf{x}'=P(t)\,\mathbf{x}$ with $P(t)$ a matrix whose entries are continuous, know how to find the Wronskian of n solutions $\mathbf{x_1}(t),\mathbf{x_2}(t),\ldots,\mathbf{x_n}(t).$ If the Wronskian of the n solutions $W\left(\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n}\right)$ is nonzero at every point of some interval I, then the solutions are linearly independent on I.
- Given the general solution, know how to use the Initial Conditions to determine the value of the constants $c_1, c_2, \ldots, c_n$.

Section 7.5: Homogeneous Linear Systems with Constant coefficients

Consider a system of n linear homogeneous equations with constant coefficients **x' = Ax** where the coefficient matrix **A** is real valued.

If all the eigenvalues of **A** are real and distinct, then there are n linearly independent eigenvectors and the general solution of the system is given by $\mathbf{x}=c_1\mathbf{v_1}e^{\lambda_1 t}+\cdots+c_n\mathbf{v_n}e^{\lambda_n t}$ where $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the eigenvalues and $\mathbf{v_1},\mathbf{v_2},\ldots,\mathbf{v_n}$ are the corresponding n linearly independent eigenvectors.

Section 7.6: Complex Eigenvalues

Consider a system of n linear homogeneous equations with constant coefficients **x' = Ax** where the coefficient matrix **A** is real valued.

If the eigenvalues are complex, then they occur in conjugate pairs and so do the associated eigenvectors.

Let $\lambda=p+q\,i$ be a complex eigenvalue and $\mathbf{v}=\mathbf{a}+i\,\mathbf{b}$ be the associated eigenvector, then, using Euler's formula, the solution can be written as

$$\mathbf{x}(t)=\mathbf{v}\,e^{\lambda t}=\mathbf{v}\,e^{pt+iqt}=(\boldsymbol{a}+i\,\boldsymbol{b})e^{pt}(\cos(q\,t)+i\sin(q\,t))=e^{pt}(\boldsymbol{a}\cos(q\,t)-\boldsymbol{b}\sin(q\,t))+i\,e^{pt}(\boldsymbol{a}\sin(q\,t)+\boldsymbol{b}\cos(q\,t))$$

Taking the real and imaginary part of $\mathbf{x}(t)$ we find the **two real valued** solutions of the system:

$$\mathbf{x_1}(t)=e^{pt}(\boldsymbol{a}\cos(q\,t)-\boldsymbol{b}\sin(q\,t)) \quad \text{and} \quad \mathbf{x_2}(t)=e^{pt}(\boldsymbol{a}\sin(q\,t)+\boldsymbol{b}\cos(q\,t))$$

It can be shown that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent and therefore they can be used to write the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

NOTE: Rather than memorizing the formulas for $\mathbf{x}_1$(t) and $\mathbf{x}_2$(t), you should be able to derive them for every specific xample using Euler's formula and separating the real and imaginary parts.