Alternating Series

A series of the form
\[
\sum_{n=1}^{\infty} (-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1}a_n + \cdots
\]
converges if
\[
0 < a_{n+1} < a_n \quad \text{for all } n \quad \text{and} \quad \lim_{n \to \infty} a_n = 0.
\]

Example. Show that the following alternating harmonic series converges:
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.
\]

Series of Both Positive and Negative Terms

Theorem: Convergence of Absolute Values Implies Convergence

If \(\sum |a_n|\) converges, then so does \(\sum a_n\).

Explain how we know that the following series converges
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \ldots.
\]

We say that the series \(\sum a_n\) is
- absolutely convergent if \(\sum a_n\) and \(\sum |a_n|\) both converge.
- conditionally convergent if \(\sum a_n\) converges but \(\sum |a_n|\) diverges.

Test for convergence.

1. \(\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}\)
2. \(\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + n^2 + 8}}{n \ln n + 4}\)
3. \(\sum_{n=2}^{\infty} \frac{n^2}{n \sin^2 n}\)
4. \(\sum_{n=1}^{\infty} \frac{n^3 + 1}{2^n + 1}\)
5. \(\sum_{n=1}^{\infty} \frac{n!}{(2n)!}\)
6. \(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n + 1}\)
Power Series

A power series about \( x = a \) is a sum of constants times powers of \((x - a)\):

\[
C_0 + C_1(x - a) + C_2(x - a)^2 + \cdots + C_n(x - a)^n + \cdots = \sum_{n=0}^{\infty} C_n(x - a)^n.
\]

A power series may converge for some values of \( x \) and not for others.

Intervals of Convergence

Each power series falls into one of the three following cases, characterized by its radius of convergence, \( R \).

- The series converges only for \( x = a \); the radius of convergence is defined to be \( R = 0 \).
- The series converges for all values of \( x \); the radius of convergence is defined to be \( R = \infty \).
- There is a positive number \( R \), called the radius of convergence, such that the series converges for \(|x - a| < R\) and diverges for \(|x - a| > R\). See Figure 9.11.
- The interval of convergence is the interval between \( a - R \) and \( a + R \), including any endpoint where the series converges.

![Diagram showing the interval of convergence](image)

Figure: Radius of convergence, \( R \), determines an interval, centered at \( x = a \), in which the series converges

\[
\begin{align*}
8. & \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 + 7} \\
9. & \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \ln n} \\
10. & \sum_{n=1}^{\infty} \frac{1}{n^2} \tan \left( \frac{1}{n} \right) \\
11. & \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2} \\
12. & \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{2^n}
\end{align*}
\]
Theorem: Method for Computing Radius of Convergence

To calculate the radius of convergence, $R$, for the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, use the ratio test with $a_n = c_n(x-a)^n$.

- If $\lim_{n \to \infty} |a_{n+1}/a_n|$ is infinite, then $R = 0$.
- If $\lim_{n \to \infty} |a_{n+1}/a_n| = 0$, then $R = \infty$.
- If $\lim_{n \to \infty} |a_{n+1}/a_n| = K|x-a|$, where $K$ is finite and nonzero, then $R = 1/K$.

Determine radius of convergence and the interval of convergence of the following power series:

1. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$
2. $\sum_{n=2}^{\infty} \frac{(-1)^n(x-2)^{2n}}{n^2}$
3. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$
4. $(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + \ldots$
5. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots$
6. $1 + 2x + 2^3x^2 + 2^4x^3 + 2^6x^4 + \ldots + 2^{2n-1}x^{2n} + \ldots$
7. $2(x+5)^3 + 3(x+5)^5 + \frac{4(x+5)^7}{2!} + \frac{5(x+5)^9}{3!} + \ldots$
8. $\sum_{n=1}^{\infty} \frac{2^n(x-1)^n}{n}$