MAT 267 Spring 2015
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Test 3 Solutions

Multiple Choice

1) Evaluate $\iiint_E ze^{2x+y}$ where E is the box $0 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 5$.

The iterated integral is written as

$$\int_0^2 \int_0^3 \int_0^5 ze^{2x+y}\ dz\ dy\ dx.$$

Since $ze^{2x+y} = ze^{2x}e^y$, the integrand is separable and we can do each of the single integrals separately:

$$\int_0^2 \int_0^3 \int_0^5 ze^{2x+y}\ dz\ dy\ dx = \int_0^2 e^{2x}\ dx \int_0^3 e^y\ dy \int_0^5 z\ dz.$$

The x integral is

$$\int_0^2 e^{2x}\ dx = \frac{1}{2}e^{2x}\Big|_{x=0}^{x=2} = \frac{1}{2}(e^4 - 1).$$

The y integral is

$$\int_0^3 e^y\ dx = e^y\Big|_{y=0}^{y=3} = (e^3 - 1).$$

The z integral is

$$\int_0^5 z\ dz = \frac{1}{2}z^2\Big|_{z=0}^{z=5} = \frac{25}{2}.$$

The triple integral is the product of these three numbers:

$$\int_0^2 \int_0^3 \int_0^5 ze^{2x+y}\ dz\ dy\ dx = \frac{25}{4}(e^4 - 1)(e^3 - 1) \approx 6393.43.$$

2) Let E be the solid region bounded by sphere of radius 4 in the first octant. Find the appropriate integral for $\iiint_E \sqrt{x^2 + y^2 + z^2}\ dV$ in spherical coordinates.

The first octant corresponds to $0 \le \phi \le \pi/2$ and $0 \le \theta \le \pi/2$. The integrand is just

$$\sqrt{x^2 + y^2 + z^2} = \rho$$

and the differential volume dV in spherical coordinates is

$$dV = \rho^2 \sin(\phi).$$

The triple integral in spherical coordinates is thus

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 \rho\ \rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 \rho^3 \sin(\phi)\ d\rho\ d\phi\ d\theta.$$

3) What are the cylindrical coordinates of the point whose rectangular coordinates are $(x, y, z) = (4, 3, 0)$?

Cylindrical coordinates have the form (r, θ, z) where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$ and $z = z$. Therefore

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5,$$
$$\theta = \tan^{-1}(3/4),$$

$$z = 0,$$

so

$$(r, \theta, z) = (5, \tan^{-1}(3/4), 0).$$

4) Find the gradient vector field for $f(x, y) = y^2 + e^{2x}$.

By definition, the gradient field is $\nabla f = \langle f_x, f_y \rangle$. The partial derivatives of f are

$$f_x = 2e^{2x}, \quad f_y = 2y.$$

Therefore the gradient field is

$$\nabla f = \langle 2e^{2x}, 2y \rangle = 2e^{2x}\mathbf{i} + 2y\mathbf{j}.$$

5) Suppose $\mathbf{F}(x, y, z)$ is a gradient field with $\mathbf{F} = \nabla f$, S is a level surface of f and C is a curve on S. What is the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$?

Since $\mathbf{F} = \nabla f$, then by the fundamental theorem of line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p})$$

where $\mathbf{p}$ and $\mathbf{q}$ are the endpoints of the curve C. Since S is a level surface on f, then for any point $\mathbf{r}$ on S, we know that $F(\mathbf{r}) = K$ for some value K. Since the curve C is on S, $\mathbf{p}$ and $\mathbf{q}$ are both on S so $f(\mathbf{p}) = f(\mathbf{q}) = K$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{q}) - f(\mathbf{p}) = K - K = 0.$$

7) Evaluate $\int_C y \, dx$ where C is the circle $x^2 + y^2 = 25$ with positive orientation.

We first parameterize C as

$$C : x(t) = 5\cos(t), \;\; y(t) = 5\sin(t), \;\; 0 \le t \le 2\pi.$$

Then

$$dx = \frac{dx}{dt} \; dt = -5\sin(t) \; dt$$

and the line integral then becomes

$$\int_C y \; dx = \int_0^{2\pi} y(t)\frac{dx}{dt} \; dt = \int_0^{2\pi} (5\sin(t))(-5\sin(t)) \; dt = -25 \int_0^{2\pi} \sin^2(t) \; dt.$$

Using the trig identity $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$, we get

$$-25 \int_0^{2\pi} \frac{1}{2}(1 - \cos(2t)) \; dt = -\frac{25}{2}(t - \frac{1}{2}\sin(2t))\Big|_{t=0}^{t=2\pi} = -\frac{25}{2}(2\pi) = -25\pi.$$

Note that we could have also done this with Green's Theorem, using $P = y$ and $Q = 0$. This would give us

$$\int_C y \; dx = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \; dA = \int_C -1 \; dA = -A(D) = -\pi(5)^2 = -25\pi,$$

where $A(D) = \pi(5)^2$ is the area of the circle enclosed by C.

Free Response

1) Let the curve C be the line segment from $(2,-1,3)$ to $(5,1,5)$ and let $\mathbf{F}(x,y,z) = \langle -y, z, x\rangle$ be a force field. Calculate the work done by $\mathbf{F}$ to move a particle along the curve C.

The work W is given by the line integral

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\ dt$$

where $\mathbf{r}(t),\ a \le t \le b$ is the parameterization of C. We parameterize the line segment from $(2,-1,3)$ to $(5,1,5)$ as

$$\mathbf{r}(t) = (1-t)\langle 2,-1,3\rangle + t\langle 5,1,5\rangle, \quad 0 \le t \le 1.$$

Simplifying this, we get

$$\mathbf{r}(t) = \langle 2-2t+5t, t-1+t, 3-3t+5t\rangle = \langle 3t+2, 2t-1, 2t+3\rangle.$$

The tangent vector of this curve is

$$\mathbf{r}'(t) = \langle 3,2,2\rangle.$$

Evaluating $\mathbf{F}$ along the curves, we get

$$\mathbf{F}(\mathbf{r}(t)) = \langle -y(t), z(t), x(t)\rangle = \langle 1-2t, 2t+3, 3t+2\rangle.$$

Therefore

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \langle 1-2t, 2t+3, 3t+2\rangle\cdot\langle 3,2,2\rangle = 3(1-2t)+2(2t+3)+2(3t+2) = 4t+13.$$

The work is therefore

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 4t+13\ dt = 2t^2+13t\Big|_{t=0}^{t=1} = 2+13 = 15.$$

2) Use Green's Theorem to evaluate $\int_C (e^{x^2}-y)\ dx + (2x+\sin^2 y)\ dy$ where C is the positively oriented circle $x^2+y^2=36$.

The general formula for Green's Theorem is

$$\int_C P\ dx + Q\ dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA$$

where D is the region enclosed by C. For this problem,

$$\frac{\partial Q}{\partial x} = 2, \quad \frac{\partial P}{\partial y} = -1,$$

so we get the double integral

$$\iint_D 2-(-1)\ dA = \iint_D 3\ dA = 3A(D)$$

where $A(D) = \pi(6)^2$ is the area of a disc of radius 6. The answer is thus

$$\iint_D\ dA = 3A(D) = 3\pi(6)^2 = 108\pi.$$

3) Let $\mathbf{F}(x, y, z) = (2xyz^3)\mathbf{i} + (x^2z^3 + \cos y)\mathbf{j} + (3x^2yz^2)\mathbf{k}$.

a) Find a potential function for $\mathbf{F}$.

The potential function $f(x, y, z)$ must satisfy $\nabla f = \mathbf{F}$, which gives is the three equations

$$f_x = 2xyz^3, \quad f_y = x^2z^3 + \cos y, \quad f_z = 3x^2yz^2.$$

Integrating the first equation with respect to x gives us

$$f(x, y, z) = x^2yz^3 + g(y, z).$$

Differentiating this with respect to y and using the second equation, we get

$$f_y = x^2z^3 + g_y = x^2z^3 + \cos y \Rightarrow g_y = \cos y \Rightarrow g(y, z) = \sin(y) + h(z).$$

The potential function is therefore

$$f(x, y, z) = x^2yz^3 + \sin y + h(z).$$

Differentiating this with respect to z and using the third equation, we get

$$f_z = 3x^2yz^2 + h'(z) = 3x^2yz^2 \Rightarrow h'(z) = 0 \Rightarrow h(z) = K,$$

where K is any constant. Therefore any function of the form

$$f(x, y, z) = x^2yz^3 + \sin y + K$$

is a potential function for $\mathbf{F}$.

b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any curve from $(2, 0, 5)$ to $(3, 2, 3)$.

By the fundamental theorem of line integrals,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{(2,0,5)}^{(3,2,3)} \nabla f \cdot d\mathbf{r} \\
&= f(3, 2, 3) - f(2, 0, 5) \\
&= [(3^2)(2)(3^3) + \sin 2 + K] - [0 + 0 + K] \\
&= 486 + \sin 2 \\
&\approx 486.9093.
\end{aligned}$$