1) Evaluate $\iiint_E ze^{2x+y} \, dz \, dy \, dx$ where $E$ is the box $0 \leq x \leq 2$, $0 \leq y \leq 3$, $0 \leq z \leq 5$.

The iterated integral is written as

$$\int_0^2 \int_0^3 \int_0^5 ze^{2x+y} \, dz \, dy \, dx.$$

Since $ze^{2x+y} = ze^{2x}e^y$, the integrand is separable and we can do each of the single integrals separately:

$$\int_0^2 \int_0^3 \int_0^5 ze^{2x+y} \, dz \, dy \, dx = \int_0^2 e^{2x} \, dx \int_0^3 e^y \, dy \int_0^5 z \, dz.$$

The $x$ integral is

$$\int_0^2 e^{2x} \, dx = \frac{1}{2}e^{2x} \bigg|_{x=0}^{x=2} = \frac{1}{2}(e^4 - 1).$$

The $y$ integral is

$$\int_0^3 e^y \, dx = e^y \bigg|_{y=0}^{y=3} = (e^3 - 1).$$

The $z$ integral is

$$\int_0^5 z \, dz = \frac{1}{2}z^2 \bigg|_{z=0}^{z=5} = \frac{25}{2}.$$  

The triple integral is the product of these three numbers:

$$\int_0^2 \int_0^3 \int_0^5 ze^{2x+y} \, dz \, dy \, dx = \frac{25}{4}(e^4 - 1)(e^3 - 1) \approx 6393.43.$$

2) Let $E$ be the solid region bounded by sphere of radius 4 in the first octant. Find the appropriate integral for $\iiint_E \sqrt{x^2 + y^2 + z^2} \, dV$ in spherical coordinates.

The first octant corresponds to $0 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq \pi/2$. The integrand is just

$$\sqrt{x^2 + y^2 + z^2} = \rho$$

and the differential volume $dV$ in spherical coordinates is

$$dV = \rho^2 \sin(\phi).$$

The triple integral in spherical coordinates is thus

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 \rho \, \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^4 \rho^3 \sin(\phi) \, d\rho \, d\phi \, d\theta.$$

3) What are the cylindrical coordinates of the point whose rectangular coordinates are $(x, y, z) = (4, 3, 0)$?

Cylindrical coordinates have the form $(r, \theta, z)$ where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$ and $z = z$. Therefore

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5,$$

$$\theta = \tan^{-1}(3/4),$$

and $z = 0$. Thus, the cylindrical coordinates are $(5, \theta, 0)$.
\( z = 0, \)

so

\((r, \theta, z) = (5, \tan^{-1}(3/4), 0).\)

4) Find the gradient vector field for \( f(x, y) = y^2 + e^{2x}. \)

By definition, the gradient field is \( \nabla f = \langle f_x, f_y \rangle. \) The partial derivatives of \( f \) are

\[ f_x = 2e^{2x}, \quad f_y = 2y. \]

Therefore the gradient field is

\[ \nabla f = \langle 2e^{2x}, 2y \rangle = 2e^{2x}i + 2yj. \]

5) Suppose \( F(x, y, z) \) is a gradient field with \( F = \nabla f, \) \( S \) is a level surface of \( f \) and \( C \) is a curve on \( S. \) What is the value of \( \int_C F \cdot dr? \)

Since \( F = \nabla f, \) then by the fundamental theorem of line integrals,

\[ \int_C F \cdot dr = \int_C \nabla f \cdot dr = f(q) - f(p) \]

where \( p \) and \( q \) are the endpoints of the curve \( C. \) Since \( S \) is a level surface on \( f, \) then for any point \( r \) on \( S, \) we know that \( F(r) = K \) for some value \( K. \) Since the curve \( C \) is on \( S, \) \( p \) and \( q \) are both on \( S \) so \( f(p) = f(q) = K. \) Therefore,

\[ \int_C F \cdot dr = f(q) - f(p) = K - K = 0. \]

7) Evaluate \( \int_C y \ dx \) where \( C \) is the circle \( x^2 + y^2 = 25 \) with positive orientation.

We first parameterize \( C \) as

\[ C : x(t) = 5\cos(t), \ y(t) = 5\sin(t), \ 0 \leq t \leq 2\pi. \]

Then

\[ dx = \frac{dx}{dt} \ dt = -5\sin(t) \ dt \]

and the line integral then becomes

\[ \int_C y \ dx = \int_0^{2\pi} y(t) \frac{dx}{dt} \ dt = \int_0^{2\pi} (5\sin(t))(-5\sin(t)) \ dt = -25 \int_0^{2\pi} \sin^2(t) \ dt. \]

Using the trig identity \( \sin^2(t) = \frac{1}{2}(1 - \cos(2t)), \) we get

\[ -25 \int_0^{2\pi} \frac{1}{2}(1 - \cos(2t)) \ dt = -\frac{25}{2} \left( t - \frac{1}{2} \sin(2t) \right) \Bigg|_{t=0}^{t=2\pi} = -\frac{25}{2}(2\pi) = -25\pi. \]

Note that we could have also done this with Green’s Theorem, using \( P = y \) and \( Q = 0. \) This would give us

\[ \int_C y \ dx = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \ dA = \int_C -1 \ dA = -A(D) = -\pi(5)^2 = -25\pi, \]

where \( A(D) = \pi(5)^2 \) is the area of the circle enclosed by \( C. \)
Let the curve \( C \) be the line segment from \((2, -1, 3)\) to \((5, 1, 5)\) and let \( \mathbf{F}(x, y, z) = \langle -y, z, x \rangle \) be a force field. Calculate the work done by \( \mathbf{F} \) to move a particle along the curve \( C \).

The work \( W \) is given by the line integral

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

where \( \mathbf{r}(t), \ a \leq t \leq b \) is the parameterization of \( C \). We parameterize the line segment from \((2, -1, 3)\) to \((5, 1, 5)\) as

\[
\mathbf{r}(t) = (1 - t)\langle 2, -1, 3 \rangle + t\langle 5, 1, 5 \rangle, \quad 0 \leq t \leq 1.
\]

Simplifying this, we get

\[
\mathbf{r}(t) = \langle 2 - 2t + 5t, t - 1 + t, 3 - 3t + 5t \rangle = \langle 3t + 2, 2t - 1, 2t + 3 \rangle.
\]

The tangent vector of this curve is

\[
\mathbf{r}'(t) = \langle 3, 2, 2 \rangle.
\]

Evaluating \( \mathbf{F} \) along the curves, we get

\[
\mathbf{F}(\mathbf{r}(t)) = \langle -y(t), z(t), x(t) \rangle = \langle 1 - 2t, 2t + 3, 3t + 2 \rangle.
\]

Therefore

\[
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle 1 - 2t, 2t + 3, 3t + 2 \rangle \cdot \langle 3, 2, 2 \rangle = 3(1 - 2t) + 2(2t + 3) + 2(3t + 2) = 4t + 13.
\]

The work is therefore

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t + 13 \, dt = 2t^2 + 13t \bigg|_{t=1}^{t=0} = 2 + 13 = 15.
\]

2) Use Green’s Theorem to evaluate \( \int_C (e^x^2 - y) \, dx + (2x + \sin^2 y) \, dy \) where \( C \) is the positively oriented circle \( x^2 + y^2 = 36 \).

The general formula for Green’s Theorem is

\[
\int_C P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA
\]

where \( D \) is the region enclosed by \( C \). For this problem,

\[
\frac{\partial Q}{\partial x} = 2, \quad \frac{\partial P}{\partial y} = -1,
\]

so we get the double integral

\[
\iint_D 2 - (-1) \, dA = \iint_D 3 \, dA = 3A(D)
\]

where \( A(D) = \pi(6)^2 \) is the area of a disc of radius 6. The answer is thus

\[
\iint_D dA = 3A(D) = 3\pi(6)^2 = 108\pi.
\]
3) Let \( \mathbf{F}(x, y, z) = (2xyz^3)i + (x^2z^3 + \cos y)j + (3x^2yz^2)k \).

a) Find a potential function for \( \mathbf{F} \).

The potential function \( f(x, y, z) \) must satisfy \( \nabla f = \mathbf{F} \), which gives us the three equations

\[
\begin{align*}
f_x &= 2xyz^3, \\
f_y &= x^2z^3 + \cos y, \\
f_z &= 3x^2yz^2.
\end{align*}
\]

Integrating the first equation with respect to \( x \) gives us

\[
f(x, y, z) = x^2yz^3 + g(y, z).
\]

Differentiating this with respect to \( y \) and using the second equation, we get

\[
f_y = x^2z^3 + g_y = x^2z^3 + \cos y \Rightarrow g_y = \cos y \Rightarrow g(y, z) = \sin(y) + h(z).
\]

The potential function is therefore

\[
f(x, y, z) = x^2yz^3 + \sin y + h(z).
\]

Differentiating this with respect to \( z \) and using the third equation, we get

\[
f_z = 3x^2yz^2 + h'(z) = 3x^2yz^2 \Rightarrow h'(z) = 0 \Rightarrow h(z) = K,
\]

where \( K \) is any constant. Therefore any function of the form

\[
f(x, y, z) = x^2yz^3 + \sin y + K
\]

is a potential function for \( \mathbf{F} \).

b) Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is any curve from \((2, 0, 5)\) to \((3, 2, 3)\).

By the fundamental theorem of line integrals,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{(2,0,5)}^{(3,2,3)} \nabla f \cdot d\mathbf{r}
\]

\[
= f(3, 2, 3) - f(2, 0, 5)
\]

\[
= [(3^2)(2)(3^3) + \sin 2 + K] - [0 + 0 + K]
\]

\[
= 486 + \sin 2
\]

\[
\approx 486.9093.
\]