MAT 267 Spring 2015
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Test 2 Solutions

1) Use the chain rule to find $\frac{\partial z}{\partial s}$ if $z = e^{xy},\ x = 7s + 8t,\ y = st^4$.

Applying the chain rule, we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (ye^{xy})(7) + (xe^{xy})(t^4) = 7ye^{xy} + xe^{xy}t^4.$$

2) Find the maximum rate of change for $f(x, y) = x^3y^2$ at the point $(3, 2)$.

The maximum rate of change at any point is the magnitude of the gradient vector at that point. The gradient vector for this function is

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 3x^2y^2, 2x^3y \rangle.$$

Evaluating this at the point $(3, 2)$ gives us

$$\nabla f(3, 2) = \langle 3(3)^2(2)^2, 2(3)^3(2) \rangle = \langle 108, 108 \rangle = 108\langle 1, 1 \rangle.$$

The maximum rate of change at $(3, 2)$ is the magnitude of this vector:

$$|\nabla f(3, 2)| = 108\sqrt{1^2 + 1^2} = 108\sqrt{2}.$$

3) Reverse the order of integration for the double integral $\int_0^1 \int_y^1 f(x, y)\ dx\ dy$.

The region of integration is the triangle bounded by $y = 0,\ y = 1,\ x = y$ and $x = 1$. This region can also be stated as

$$D = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le x\}.$$

Therefore the reversed double integral is

$$\int_0^1 \int_0^x f(x, y)\ dy\ dx.$$

4) By changing to polar coordinates, evaluate the integral $\iint_D (x^2 + y^2)^{3/2}\ dA$ where $D$ is the disc $x^2 + y^2 \le 4$.

The disc $D$ in polar coordinates is the polar rectangle $D = [0, 2] \times [0, 2\pi]$. The integrand in polar coordinates is

$$(x^2 + y^2)^{3/2} = (r^2)^{3/2} = r^3.$$

Therefore the double integral in polar coordinates is

$$\iint_D (x^2 + y^2)^{3/2}\ dA = \int_0^{2\pi} \int_0^2 r^3\ r\ dr\ d\theta = \int_0^{2\pi} \int_0^2 r^4\ dr\ d\theta.$$

This integral is separable so we can do each single integral separately

$$\int_0^{2\pi} \int_0^2 r^4\ dr\ d\theta = \left[\int_0^{2\pi} d\theta\right]\left[\int_0^2 r^4\ dr\right] = (2\pi) \cdot \frac{1}{5}r^5\Big|_{r=0}^{r=2} = (2\pi) \cdot \frac{32}{5} = \frac{64\pi}{5}.$$

5) Find the differential of the function $z = 3y\sqrt{x}$.

The differential for a function of two variables $z = f(x, y)$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{3y}{2\sqrt{x}}dx + 3\sqrt{x}dy.$$

6) Find the domain of the function $f(x, y) = \sqrt{x} + \sqrt{y}$.

Since $\sqrt{x}$ and $\sqrt{y}$ both appear in the function, we must have that $x \geq 0$ and $y \geq 0$. Geometrically, this is the first quadrant of $\mathbb{R}^2$. We could also state it in set notation as $D = \{(x, y) \mid x \geq 0,\ y \geq 0\}$.

7) Find the directional derivative of the function $f(x, y) = 2x^2y^3 + 3x$ at the point $(1, -2)$ in the direction of the vector $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$.

The general formula for the directional derivative at the point $\mathbf{r}$ in the direction of $\mathbf{v}$ is

$$D_{\mathbf{v}}f(\mathbf{r}) = \nabla f(\mathbf{r}) \cdot \hat{\mathbf{v}}$$

where $\hat{\mathbf{v}}$ is the unit vector parallel to $\mathbf{v}$. The gradient of $f$ is

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 4xy^3 + 3, 6x^2y^2 \rangle.$$

Evaluating this at $(1, -2)$ gives us

$$\nabla f(1, -2) = \langle 4(1)(-2)^3 + 3, 6(1)^2(-2)^2 \rangle = \langle -29, 24 \rangle.$$

The unit vector parallel to $\mathbf{v}$ is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 5, 12 \rangle}{\sqrt{5^2 + 12^2}} = \frac{1}{13}\langle 5, 12 \rangle.$$

The directional derivative is thus

$$D_{\mathbf{v}}f(\mathbf{r}) = \nabla f(\mathbf{r}) \cdot \hat{\mathbf{v}} = \langle -29, 24 \rangle \cdot \frac{1}{13}\langle 5, 12 \rangle = \frac{1}{13}(-29 \cdot 5 + 24 \cdot 12) = \frac{143}{13} = 11.$$

8) Find an equation of the tangent plane to the surface $xyz + y^2 + z^3 = 6$ at the point $(1, 2, 3)$.

Notice that this surface is of the form $F(x, y, z) = c$ where $F(x, y, z) = xyz + y^2 + z^3$ and is therefore a level surface of $F$. Therefore the normal vector of the tangent plane at $(1, 2, 3)$ is the gradient of $F$ evaluated at this point. The gradient of this function is

$$\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle = \langle yz, xz + 2y, xy + 3z^2 \rangle.$$

Evaluating this at $(1, 2, 3)$ gives us the normal vector

$$\mathbf{n} = \nabla F(1, 2, 3) = \langle (2)(3), (1)(3) + 2(2), (1)(2) + 3(3)^2 \rangle = \langle 6, 7, 29 \rangle.$$

The equation of the tangent plane is therefore

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \Rightarrow \langle 6, 7, 29 \rangle \cdot \langle x - 1, y - 2, z - 3 \rangle = 0$$

$$\Rightarrow 6(x - 1) + 7(y - 2) + 29(z - 3) = 0.$$

9) Find all critical points for the function $f(x, y) = x^3 - 12xy + 8y^3$ and classify them as either a local minimum, local maximum or saddle point.

Setting each partial derivative equal to zero gives

$$f_x = 3x^2 - 12y = 0, \quad f_y = -12x + 24y^2 = 0$$

which is a system of equations that we must solve. Solving for $x$ in the second equation gives us $x = 2y^2$. Inserting this into the first equation, we get

$$3(2y^2)^2 - 12y = 0 \Rightarrow 12y^4 - 12y = 0 \Rightarrow 12y(y^3 - 1) = 0$$

so $y = 0$ and $y = 1$ are both solutions. If $y = 0$ then the system of equations reduces to

$$3x^2 = 0, \quad -12x = 0$$

so $x = 0$ is the only solution that satisfies both equations. Therefore $(0, 0)$ is a critical point. If $y = 1$ then the system of equations reduces to

$$3x^2 - 12 = 0, \quad -12x + 24 = 0$$

The first equation has solutions $x = \pm 2$ and the second equation has only one solution $x = 2$. Therefore $x = 2$ is the only solution that satisfies both equations and $(2, 1)$ is a critical point. To classify the critical points, we compute all of the second derivatives:

$$f_{xx} = 6x, \quad f_{yy} = 48y, \quad f_{xy} = -12.$$

The discriminant is thus

$$D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(48y) - (-12)^2 = 288xy - 144 = 144(2xy - 1).$$

We now apply the second derivative test for each critical point:

- Test for $(0, 0)$: $D(0, 0) = -144 < 0 \Rightarrow (0, 0)$ is a saddle point.
- Test for $(2, 1)$: $D(2, 1) = 144(3) > 0$, $f_{xx}(2, 1) = 12 > 0 \Rightarrow (2, 1)$ is a local minimum.

10) Compute the double integral $\iint_D 2x + 3y^2 \; dy \; dx$ where $D = [0, 1] \times [-1, 2]$.

The iterated integral is stated as

$$\int_0^1 \int_{-1}^2 2x + 3y^2 \; dy \; dx.$$

Doing the inner integral with respect to $y$, we get

$$\int_{-1}^2 2x + 3y^2 \; dy = 2xy + y^3 \Big|_{y=-1}^{y=2} = [2x(2) + 2^3] - [2x(-1) + (-1)^3] = (4x + 8) - (-2x - 1) = 6x + 9.$$

Plugging this into the outer integral gives

$$\int_0^1 6x + 9 \; dx = 3x^2 + 9x \Big|_{x=0}^{x=1} = 3 + 9 = 12.$$