Practice for Test 3, MAT 266

Determine whether the following series converges absolutely. Justify your answer with the proper series test.

1. $\sum_{n=0}^{\infty} \frac{(-1)^n (n+5)!}{5!\,n!\,4^n}$ 2. $\sum_{n=1}^{\infty} \frac{(-2)^{n+3} n^3}{3^n}$ 3. $\sum_{n=0}^{\infty} \frac{n!}{1{,}000{,}000^n}$ 4. $\sum_{n=0}^{\infty} \frac{(2n)!}{3^n (n!)^2}$

Determine the radius of convergence and the largest open interval of convergence. Justify your answer with the proper series test.

5. $\sum_{n=0}^{\infty} \frac{(-1)^n 6^n x^{2n}}{n!}$ 6. $\sum_{n=0}^{\infty} \frac{5^n x^n (n+4)}{(n^2+16)}$ 7. $\sum_{n=1}^{\infty} \frac{n^2 (x+4)^n}{7^n \sqrt{2n-1}}$ 8. $\sum_{n=1}^{\infty} \frac{(-1)^n n (x-5)^n}{3^{2n}(n+1)}$

9. For the function $f(x) = \frac{3}{16+x^2}$
   (a) Find the power series representation of $f(x)$.
   (b) Determine the radius and interval of convergence for the power series in part (a).

10. For the function $f(x) = \frac{1}{6+x}$
   (a) Find the power series representation of $f(x)$.
   (b) Determine the radius and interval of convergence for the power series in part (a).
   (c) Use part (a) to find the power series representation of $g(x) = \ln(6+x)$ along with its radius of convergence.

11. The function $f(x) = \frac{x^2}{1-2x}$ is represented by the power series in x, $f(x) = \sum_{n=0}^{\infty} c_n x^n$. Find the coefficients: $c_0, c_1, c_2, c_3, c_4, c_5,$ and c_6.

12. Write a partial sum for the power series in x which represents the function $f(x) = x\ln(1+3x)$ consisting of the first four nonzero terms. State the radius of convergence.

13. For the function $f(x) = x\cos(x^3)$
   a. Find the partial sum of the Maclaurin series for $f(x)$ consisting of the first four nonzero terms.
   b. Use the answer in part (a) to find the first four nonzero terms of the partial sum of the Maclaurin series for $F(x) = \int x\cos(x^3)\,dx$ with $F(0) = 0$.
   c. Use the above to estimate the definite integral $\int_0^1 x\cos(x^3)\,dx$. Round your answer to 8 decimal places.

14. Let $F(x) = \int e^{-x^2}\,dx$ with $F(0) = 0$.
   a. Find the partial sum of the Maclaurin series for $F(x)$ consisting of the first four nonzero terms.
   b. Use the answer in part (a) to estimate the definite integral $\int_0^{1/2} e^{-x^2}\,dx$. Round your answer to 6 decimal places.

15. The function $f(x) = \sin\left(\frac{2x}{3}\right)$ has a Maclaurin series. Find a formula for the Maclaurin series.

16. The function $f(x) = e^{-2x}$ has a Maclaurin series. Find the first four terms of the series.

17. Find the first four terms of the Taylor series of $f(x) = \sqrt{x}$ centered at $a = 16$. Use that result to approximate $\sqrt{15.2}$.

18. Find the first four terms of the Taylor series of $f(x) = \sqrt[3]{x}$ centered at $a = 8$. Use that result to approximate $\sqrt[3]{9}$.

19. Find the first four terms of the Taylor series of $f(x) = \ln(x)$ centered at $a = 5$.

20. Find the first four terms of the Taylor series of $f(x) = \cos(x)$ centered at $a = \frac{\pi}{3}$.

21. Use Maclaurin series to find the following limits:

    a. $\lim_{x \to 0} \frac{\sin(x^2) - x^2}{x^6}$ b. $\lim_{x \to 0} \frac{x(\tan^{-1}(x) - x)}{\cos(x) - e^{-x^2/2}}$

22. Eliminate the parameter to find the Cartesian equation for $x = 1 + t,\ y = t^2 + 2$.
23. Eliminate the parameter to find the Cartesian equation for $x = 2\sin(t),\ y = 3\cos(t)$.
24. Eliminate the parameter to find the Cartesian equation for $x = \sec^2(t),\ y = 2\tan(t)$.
25. Eliminate the parameter to find the Cartesian equation for $= 1 + 2\cos(t),\ y = -2 + 2\sin(t)$.
26. Suppose parametric equations for the line segment between (1, 2) and (4, 7) have the form $x = a + bt$, $y = c + dt$. Find a, b, c and d so that the parametric curve starts at $(1, 2)$ when $t = 0$, and ends at $(4, 7)$ when $t = 1$.
27. Find parametric equations for the path of a particle that moves along the circle $x^2 + y^2 = 25$ twice around clockwise, $0 \le t \le 2\pi$, starting at (5,0).
28. Find the slope of the line tangent to the parametric curve $x = t + \cos(\pi t)$, $y = t + \sin(\pi t)$ at $t = \frac{1}{2}$.
29. Find an equation of the line tangent to the parametric curve $x = \cos^3(t)$, $y = \sin^3(t)$, at $t = \frac{\pi}{3}$.
30. For the parametric curve $x = \frac{t^3}{9} - t,\ y = \frac{t^4}{12} - t^2 + 1$
    a. Find an equation of the line tangent to the curve at $t = 3$.
    b. Find all values of t where the tangent is horizontal or vertical.
31. For the parametric curve $x = e^{2t} - t - 3,\ y = -t^3 + 2t$
    a. Find an equation of the line tangent to the curve at $t = 0$.
    b. Find the exact value of t corresponding to the leftmost point of the curve, and the value(s) of t where the tangent is horizontal.
32. Find the arc length of the parametric curve $x = \cos^3(t),\ y = \sin^3(t),\ 0 \le t \le \frac{\pi}{2}$.
33. Find the exact arc length of the loop of the parametric curve, $x = t - \frac{t^3}{3}$, $y = t^2, -\sqrt{3} \le t \le \sqrt{3}$.
34. Find the exact arc length of the parametric curve $x = \cos(t) + t\sin(t), y = \sin(t) - t\cos(t), 0 \le t \le \pi$.
35. Find area under the parametric curve $x = t^3 - 1,\ y = 2 + 3t - t^3,\ -1 \le t \le 2$.
36. Find area under the parametric curve $x = t - \frac{1}{t},\ y = t^2 + 1,\ \frac{1}{2} \le t \le 2$.
37. Find area under the parametric curve $x = \sin(t),\ y = 2\sin(t)\cos(t)$, $0 \le t \le \frac{\pi}{2}$.

ANSWER KEY:

1. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$, thus, by the Ratio Test, the series Converges Absolutely.
2. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3}$, thus, by the Ratio Test, the series Converges Absolutely.
3. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, thus, by the Ratio Test, the series diverges.
4. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{4}{3}$, thus, by the Ratio Test, the series diverges.
5. $I = (-\infty, \infty)$; $R = \infty$.
6. $I = \left(-\frac{1}{5}, \frac{1}{5}\right)$; $R = \frac{1}{5}$.
7. $I = (-11, 3)$; $R = 7$.
8. $I = (-4, 14)$; $R = 9$.
9. (a) $f(x) = \frac{3}{16} - \frac{3x^2}{256} + \frac{3x^4}{4096} - \frac{3x^6}{65536} + \cdots + \frac{(-1)^n 3(x)^{2n}}{16^{n+1}} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3(x)^{2n}}{16^{n+1}}$

   (b) $R = 4$, $I = (-4, 4)$
10. (a) $f(x) = \frac{1}{6} - \frac{x}{36} + \frac{x^2}{216} - \frac{x^3}{1296} + \cdots + \frac{(-1)^n x^n}{6^{n+1}} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{6^{n+1}}$

    Note: for $f(x)$, $c_0 = \frac{1}{6}$, $c_1 = -\frac{1}{36}$, $c_2 = \frac{1}{216}$, and $c_3 = -\frac{1}{1296}$.

    (b) $R = 6$, $I = (-6, 6)$

    (c) $g(x) = \ln(6) + \frac{x}{6} - \frac{x^2}{72} + \frac{x^3}{648} - \frac{x^4}{5184} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)6^{n+1}} + \cdots = \ln(6) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n6^n}$; $R = 6$

    Note: for $g(x)$, $c_0 = \ln(6)$, $c_1 = \frac{1}{6}$, $c_2 = -\frac{1}{72}$, $c_3 = \frac{1}{648}$, and $c_4 = \frac{1}{5184}$.
11. $c_0 = 0, c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 4, c_5 = 8,$ and $c_6 = 16$.
12. $f(x) \approx 3x^2 - \frac{9x^3}{2} + 9x^4 - \frac{81x^5}{4}$; $R = \frac{1}{3}$
13. (a) $f(x) \approx x - \frac{x^7}{2} + \frac{x^{13}}{24} - \frac{x^{19}}{720}$ (b) $F(x) \approx \frac{x^2}{2} - \frac{x^8}{16} + \frac{x^{14}}{336} - \frac{x^{20}}{14400}$ (c) 0.44040675
14. (a) $F(x) \approx x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}$ (b) 0.461272
15. $\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{3^{2n+1}(2n+1)!}$
16. $f(x) \approx 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3$
17. $f(x) \approx 4 + \frac{1}{8}(x-16) - \frac{1}{512}(x-16)^2 + \frac{1}{16384}(x-16)^3$; $\sqrt{15.2} = f(15.2) \approx 3.89871875$

18. $f(x) \approx 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3$; $\sqrt[3]{9} = f(9) \approx 2.080102$

19. $f(x) \approx \ln(5) + \frac{1}{5}(x-5) - \frac{1}{50}(x-5)^2 + \frac{1}{375}(x-5)^3$

20. $f(x) \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3$

21. (a) $-\frac{1}{6}$ (b) 4

22. $y = x^2 - 2x + 3$

23. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

24. $x = \frac{y^2}{4} + 1$

25. $(x-1)^2 + (y+2)^2 = 4$

26. $x = 1 + 3t,\ y = 2 + 5t$

27. $x = 5\cos(2t),\ y = -5\sin(2t)$

28. $\frac{1}{1-\pi}$

29. $y = -\sqrt{3}\,x + \frac{\sqrt{3}}{2}$

30. (a) $y = \frac{3}{2}x - \frac{5}{4}$ (b) Horizontal tangent at $t = -\sqrt{6},\ 0,\ \sqrt{6}$; Vertical tangent at $t = -\sqrt{3},\ \sqrt{3}$

31. (a) $y = 2x + 4$ (b) leftmost value at $t = \frac{1}{2}\ln\left(\frac{1}{2}\right)$; Horizontal tangent at $t = -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}$.

32. $\frac{3}{2}$

33. $4\sqrt{3}$

34. $\frac{\pi^2}{2}$

35. $\frac{81}{4}$

36. $\frac{57}{8}$

37. $\frac{2}{3}$

