MAT 210 Exam Final Review Questions

The final exam is comprehensive; it will emphasize on the last part (anti-derivative). You need to review the review problems for exam 1, and 2. In addition, below are some review questions related to the sections covered after Exam 2.

Indefinite Integral (Section 13.1)

1. Find the indefinite integrals.
   (a) $\int 2x^4 - 4x^{-2} + 5x^{-5} + 3\ dx$
   (b) $\int \frac{7}{x} + \frac{1}{3x^7} dx$
   (c) $\int \frac{2}{x^2} - 5\sqrt{x}\ dx$
   (d) $\int e^x - x^{-0.3}\ dx$
   (e) $\int (x+3)(x-2) dx$
   (f) $\int \frac{x^2+5x-2}{x} dx$

   ***Answer*:**

   (a) $\frac{2}{5}x^5 + \frac{4}{x} - \frac{5}{4x^4} + 3x + C$

   (b) $7\ln|x| - \frac{1}{18x^6} + C$

   (c) $-\frac{2}{x} - \frac{10}{3}x^{\frac{3}{2}} + C$

   (d) $e^x - \frac{x^{0.7}}{0.7} + C = e^x - \frac{10}{7}x^{0.7} + C$

   (e) $\int x^2 + x - 6\ dx = \frac{x^3}{3} + \frac{x^2}{2} - 6x + C$

   (f) $\int x + 5 - \frac{2}{x}\ dx = \frac{x^2}{2} + 5x - 2\ln|x| + C$

2. Find $f(x)$ if $f(0) = -1$ and the derivative $f'(x) = 9e^x + 9$.

   Answer: $f(x) = 9e^x + 9x - 10$

3. The velocity of a particle moving in a straight line is $v(t) = t^2 + 6$. Find the expression for the position, $s(t)$, of the particle at time t, if $s(3) = 0$.

   Answer: $s(t) = \frac{1}{3}t^3 + 6t - 27$

4. Suppose the function $C(x)$ gives the total cost (in dollars) of producing x units of a certain product. The marginal cost of producing the xth unit is $C'(x) = 0.5x + \frac{1}{x}$. If the cost to produce the first unit is 5 dollars, find the cost function $C(x)$.

Answer: $C(x) = 0.25x^2 + \ln|x| + 4.75$ dollars

$$C(x) = \int C'(x)\,dx = \int 0.5x + \frac{1}{x}dx = 0.25x^2 + \ln|x| + K$$

$C(1) = 5$, So $5 = 0.25 \cdot 1^2 + \ln|1| + K$. Then solve for constant K: $K = 5 - 0.25 = 4.75$.

Substitution (Section 13.2)

5. Use integration by substitution to find the integrals.
   (a) $\int 16e^{-3x}\,dx$ (can also use short-cut formula)
   (b) $\int (5x-2)^3 dx$ (can also use short-cut formula)
   (c) $\int \frac{1}{2x-5}dx$ (can also use short-cut formula)
   (d) $\int 4xe^{x^2-3}dx$
   (e) $\int x(x^2+1)^{10}\,dx$
   (f) $\int 15x\sqrt{-x^2+7}\,dx$
   (g) $\int (3x^2+1)(x^3+x-2)^9 dx$

   ***Answer*:**

   (a) $16 \cdot \frac{e^{-3x}}{-3} + C = -\frac{16}{3}e^{-3x} + C$
   (b) $\frac{(5x-2)^4}{4} \cdot \frac{1}{5} + C = \frac{1}{20}(5x-2)^4 + C$
   (c) $\frac{1}{2}\ln|2x-5| + C$
   (d) $2e^{x^2-3} + C$
   (e) $\frac{1}{22}(x^2+1)^{11} + C$
   (f) $-5(-x^2+7)^{\frac{3}{2}} + C$
   (g) $\frac{1}{10}(x^3+x-2)^{10} + C$

Fundamental Theorem of Calculus; Definite Integral; Left Riemann Sum (Sections 13.3, 13.4)

6. Evaluate the definite integrals.
   (a) $\int_0^1 (6x^5 + 15x^4 - 9x^2 + 1)\,dx$
   (b) $\int_2^7 \left(x + \frac{5}{x}\right)dx$
   (c) $\int_1^{10} \frac{1}{x^2}dx$
   (d) $\int_0^6 e^{-x+6}\,dx$
   (e) $\int_{-1}^1 5e^{3x}dx$
   (f) $\int_{e^3}^{e^5} \frac{2}{x}dx$

(g) $\int_{\ln 3}^{\ln 5} e^{2x} dx$

Answer:

(a) 2
(b) $\frac{45}{2} + 5\ln\left(\frac{7}{2}\right)$
(c) $\frac{9}{10}$
(d) $-1 + e^6$
(e) $\frac{5}{3}(e^3 - e^{-3})$
(f) 4
(g) 8

7. Assume that b is a positive number, solve the following equation for b.

$\int_2^b (2x-4)\, dx = 9$

Answer: $b = 5$

8. Calculate the left Riemann sum for the function $f(x) = 3x^2 + 2x - 3$ over the interval $[1, 3]$, with $n = 5$.

Answer: 22.56

$\Delta x = \frac{b-a}{n} = \frac{3-1}{5} = 0.4$, $x_0 = a = 1, x_1 = x_0 + \Delta x = 1.4, x_2 = 1.8,\ x_3 = 2.2,\ x_4 = 2.6.$
$\text{LRS} = \Delta x \cdot \big(f(1) + f(1.4) + f(1.8) + f(2.2) + f(2.6)\big) = 0.4(2 + 5.68 + 10.32 + 15.92 + 22.48) = 22.56$

9. Use a left Riemann sum to estimate the definite integral with $n = 4$ subintervals.

$$\int_2^3 \frac{1}{1+2x} dx$$

Answer: 0.18

$\Delta x = 0.25$, $\text{LRS} = 0.25\left(\frac{1}{1+2(2)} + \frac{1}{1+2(2.25)} + \frac{1}{1+2(2.5)} + \frac{1}{1+2(2.75)}\right) = 0.25(0.2 + 0.18 + 0.17 + 0.15) = 0.18$

Applications of Definite Integrals (Section 13.4)

10. A particle moves in a straight line with velocity $v(t) = -t^2 + 8$ meters per second, where t is time in seconds. Find the displacement of the particle between $t = 2$ and $t = 6$ seconds.

Answer: -37 meters

Displacement $= s(6) - s(2) = \int_2^6 v(t)\, dt = \int_2^6 (-t^2 + 8)\, dt = -\frac{112}{3} \approx -37$ meters.

11. The marginal revenue of the xth box of flash cards sold is $500e^{-0.001x}$ dollars. Find the revenue generated by selling box 101 through 5,000.

Answer: 448,598 dollars

Total revenue generated $= R(5000) - R(101) = \int_{101}^{5000} MR\, dx = \int_{101}^{5000} 500e^{-0.001x}\, dx \approx 448597.54$ dollars

12. Since YouTube first became available to the public in mid-2005, the rate at which video has been uploaded to this site can be approximated by $f(t) = 1.1t^2 - 2.6t + 2.3$ million hours of videos per year ($0 \le t \le 9$), where t is time in years since June 2005. Use a definite integral to estimate the total number of hours of video uploaded from June 2007 to June 2010.

Answer: 23 million hours of video

Total number of hours $= \int_2^5 f(t)\, dt = \int_2^5 (1.1t^2 - 2.6t + 2.3)\ dt \approx 23$ million hours of video

13. Calculate the area of the region bounded by $y = \sqrt{x}$, the x-axis, and the lines $x = 0$ and $x = 16$.

Answer: $\frac{128}{3}$

Area under curve $= \int_0^{16} \sqrt{x}\, dx = \frac{128}{3}$

Integration by Parts (Section 14.1)

Integration by parts formula: $\int u dv = uv - \int v du$

14. Use integration by parts to find the integrals.

(a) $\int 2xe^x dx$

(b) $\int (3x + 4)e^{-5x} dx$

(c) $\int \ln x\, dx$

(d) $\int x^2 \ln x\, dx$

***Answer*:**

(a) Let $u = 2x, dv = e^x dx$. Then $du = 2dx$ and $v = e^x$.
Using the formula: $\int u dv = uv - \int v du$ to get

$$\int 2xe^x dx = 2xe^x - \int e^x\, 2dx = 2xe^x - 2e^x + C$$

(b) $-\frac{1}{5}(3x + 4)e^{-5x} - \frac{3}{25}e^{-5x} + C = \left(-\frac{3}{5}x - \frac{23}{25}\right)e^{-5x} + C$
(Let $u = 3x + 4, dv = e^{-5x}dx$. Then $du = 3dx$ and $v = -\frac{1}{5}e^{-5x}$)

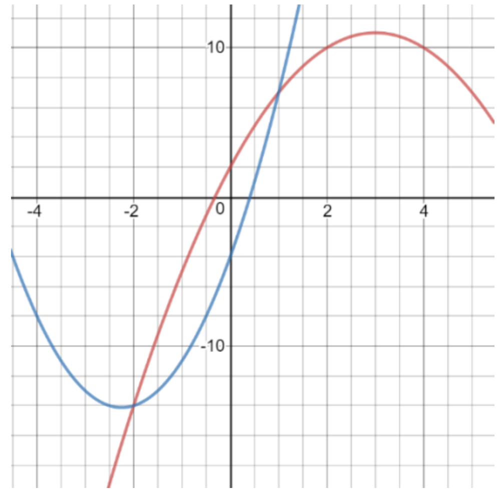
(c) $x \ln x - x + C$ (Let $u = \ln x\, , dv = dx$)

(d) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$ (Let $u = \ln x\, , dv = x^2 dx$)

Area between Curves (Section 14.2)

15. Find the area of the region enclosed by the curves of $y = -x^2 + 6x + 2$ and $y = 2x^2 + 9x - 4$.

Answer: 13.5



Find the intersection points: $-x^2 + 6x + 2 = 2x^2 + 9x - 4$

$$0 = 3x^2 + 3x - 6$$

$$0 = 3(x + 2)(x - 1)$$

So $x = -2$ and $x = 1$.

The area enclosed by the curves from -2 to 1 is

$$\int_{-2}^{1}(\text{top} - \text{bottom})\,dx = \int_{-2}^{1}[(-x^2 + 6x + 2) - (2x^2 + 9x - 4)]\,dx$$

$$= \int_{-2}^{1}(-3x^2 - 3x + 6)\,dx$$

$$= -x^3 - \frac{3}{2}x^2 + 6x|_{-2}^{1}$$

$$= (-1^3 - \frac{3}{2}1^2 + 6(1)) - \left(-(-2)^3 - \frac{3}{2}(-2)^2 + 6(-2)\right) = 13.5$$

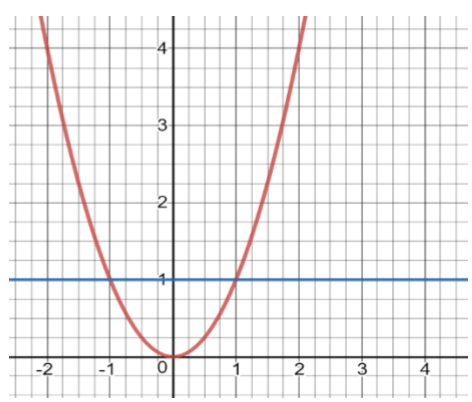
16. Find the area of the region enclosed by the curves of $f(x) = x^2 - x + 5$ and $g(x) = x + 8$.

Answer: $\frac{32}{3}$

17. Find the area of the region between $y = x^2$ and $y = -1$ from $x = -1$ and $x = 1$.

Answer: $\frac{8}{3}$

18. Which of the following calculates the area of the region(s) between the curves $y = x^2$ and $y = 1$ from $x = -1$ to $x = 2$?



A. $\int_{-1}^{2}(x^2 - 1)dx$

B. $\int_{-1}^{2}(1 - x^2)dx$

C. $\int_{-1}^{1}(1 - x^2)dx + \int_{1}^{2}(x^2 - 1)dx$

D. $\int_{-1}^{1}(x^2 - 1)dx + \int_{1}^{2}(1 - x^2)dx$

E. None of the above.

Answer: C

Average Value (Section 14.3)

19. Find the average value of $f(x) = 6e^{0.5x}$ over the interval $[-1, 3]$.

Answer: $3(e^{1.5} - e^{-0.5})$

The average value of a continuous function $f(x)$ over interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\frac{1}{3-(-1)}\int_{-1}^{3} 6e^{0.5x}\,dx = \frac{1}{4}\cdot 6\cdot\frac{e^{0.5x}}{0.5}|_{-1}^{3} = 3e^{0.5x}|_{-1}^{3} = 3(e^{1.5} - e^{-0.5})$$

20. Find the average of the function $f(x) = x^3 - x$ over the interval $[0, 2]$.

Answer: 1

21. Find the average value of the function $f(x) = 6x^2 - 4x + 7$ over the interval $[-2, 2]$.

Answer: 15

Consumer's Surplus and Producer's Surplus (section 14.4)

22. Your shop can sell 50 "I love calculus" t-shirt at $20 each per day. You decided to drop the price $1.50 per shirt and this results in 3 more t-shirts sold per day.

a. Write a linear function for the unit price p of the t-shirt sold daily as a function of q (demand), the number t-shirts can be sold at unit price p.

Answer: When unit price $p = 20$, the demand $q = 50$ and when the unit price $p = 18.5$, the demand $q = 53$. The slope is $\frac{-1.5}{3} = -\frac{1}{2}$ dollar per t-shirt. Thus, $p - 20 = -\frac{1}{2}(q - 50)$, that is $p = -\frac{1}{2}q + 45$.

b. Calculate the consumers' surplus when the unit price is $\bar{p}$=15 dollars per shirt using the demand equation found in part (a). Consumers' Surplus is defined as $\int_0^{\bar{q}} (D(q) - \bar{p})dq$.

Answer: When $\bar{p} = 15$, then $15 = -\frac{1}{2}q + 45$, $-30 = -\frac{1}{2}q$, $\bar{q} = 60$.

$$CS = \int_0^{60}((-\frac{1}{2}q + 45) - 15)dq = \int_0^{60}(-\frac{1}{2}q + 30)dq = (-\frac{1}{4}q^2 + 30q)|_0^{60}$$

$$= -\frac{1}{4}\cdot 60^2 + 30\cdot 60 = -900 + 1800 = \$\,900.$$

The consumers' surplus is 900 dollars for the first 60 t-shirt sold.

23. Calculate the producers' surplus for the supply equation at the indicated unit price $\bar{p} = \$\,160$ (Round your answer to the nearest cent.)

Producers" Surplus is defined a $\int_0^{\bar{q}} (\bar{p} - S(q))dq$ where $p = 130 + e^{0.01q}$, $\bar{p} = 160$.

Answer: When $\bar{p} = 160$, then $160 = 130 + e^{0.01q}$, $30 = e^{0.01q}$, $\bar{q} = \frac{\ln 30}{0.01} = 340.1197382$.

$$PS = \int_0^{\frac{\ln 30}{0.01}}(160 - (130 + e^{0.01q})) = \int_0^{\frac{\ln 30}{0.01}}(30 - e^{0.01q})dq = \left(30q - \frac{1}{0.01}e^{0.01q}\right)|_0^{\frac{\ln 30}{0.01}}$$

$$= \left(30 \cdot \frac{\ln 30}{0.01} - \frac{1}{0.01} \cdot e^{0.01 \cdot \frac{\ln 30}{0.01}}\right) - \left(30 \cdot 0 - \frac{1}{0.01} e^{0.01 \cdot 0}\right) = \left(30 \cdot \frac{\ln 30}{0.01} - \frac{30}{0.01}\right) + \frac{1}{0.01} = 7303.59.$$

The producers' surplus is \$ 7303.59 for the first 340 items sold.

24. A company finds that the demand for their new product is given by $p = 13 - q^{\frac{1}{4}}$, where p is the price per item and q is the number of items that can be sold per week at unit price p. The company is prepared to sell $q = \left(\frac{p-4}{2}\right)^4$ items per week at a unit price p. Find the equilibrium price $\bar{p}$ and the consumers' and producers' surpluses at the equilibrium price. What is the total social gain at the equilibrium price?

***Answer*:** To find the equilibrium price solve the following system of equations $p = 13 - q^{\frac{1}{4}}$ and $q = \left(\frac{p-4}{2}\right)^4$ simultaneously, that is find when the supply meets the demand. Substitute $q = \left(\frac{p-4}{2}\right)^4$ for q into $p = 13 - q^{\frac{1}{4}}$.

Using the identity $\left(a^k\right)^n = a^{k \cdot n}$, we obtain $p = 13 - q^{\frac{1}{4}} = 13 - \left(\frac{p-4}{2}\right)^{4 \cdot \frac{1}{4}} = 13 - \left(\frac{p-4}{2}\right)$. Solve the equation $p = 13 - \left(\frac{p-4}{2}\right)$ for p. Multiplying by 2, we get $2p = 26 - (p-4) = 30 - p$. Then $3p = 30$ and $\bar{p} = \$\,10$.

To find the corresponding demand $\bar{q}$, substitute $\bar{\boldsymbol{p}} = \mathbf{10}$ into $q = \left(\frac{p-4}{2}\right)^4$ and solve for q. Thus $\bar{\boldsymbol{q}} = \mathbf{81}$.

The equilibrium price is $\bar{\boldsymbol{p}} = \mathbf{10}$.

$$CS = \int_0^{81}\left(\left(13 - q^{\frac{1}{4}}\right) - 10\right)dq = \int_0^{81}\left(3 - q^{\frac{1}{4}}\right)dq = \left(3q - \frac{1}{1.25} q^{1.25}\right)\Big|_0^{81} = 3 \cdot 81 - \frac{1}{1.25} \cdot 81^{1.25} = \$\,48.6$$

The consumers' surplus for the first 81 items is $\mathbf{48.6}$ dollars.

Before we find the producers' surplus, we need to solve the supply equation for p. The 4th root of both sides of the equation $q = \left(\frac{p-4}{2}\right)^4$ to obtain $q^{\frac{1}{4}} = \left(\frac{p-4}{2}\right)$. Multiplying by 2 and adding 4 to both sides of the equation, we get $p = 2q^{1/4} + 4$. Let us calculate the producer surplus.

$$PS = \int_0^{81}\left(10 - \left(2q^{1/4} + 4\right)\right) = \int_0^{81}\left(6 - 2\,q^{1/4}\right)dq = \left(6q - \frac{2}{1.25} q^{1.25}\right)\Big|_0^{81} = 6 \cdot 81 - \frac{2}{1.25} 81^{1.25} = \$\,97.2$$

The producers' surplus for the first 81 items is $\mathbf{97.2}$ dollars.

The total social gain is the sum of CS and PS, that is $\mathbf{48.6 + 97.2 = 145.8}$ dollars for the first 81 items.

25. Given that x is number of items, the demand function is $d(x) = 200 - 0.2x$, and the supply function is $s(x) = 0.3x$.

    a) Find the equilibrium quantity.
    b) Find the consumers' surplus and producers' surplus at the equilibrium quantity.

    ***Answer*:** a) $x = 400$ items b) $CS = \$16{,}000$; $PS = \$24{,}000$

26. Given that x is number of items, the demand function is $d(x) = 270.4 - 0.1x^2$, and the supply function is $s(x) = 0.3x^2$.

a) Find the equilibrium quantity.
b) Find the consumers' surplus and producers' surplus at the equilibrium quantity.

Answer: a) $x = 26$ items b) $CS = \$1{,}171.73$; $PS = \$3{,}515.2$

27. Given that x is number of items, the demand function is $d(x) = \frac{2304}{\sqrt{x}}$, and the supply function is $s(x) = 9\sqrt{x}$.

a) Find the equilibrium quantity.
b) Find the consumers' surplus and producers' surplus at the equilibrium quantity.

Answer: a) $x = 256$ items b) $CS = \$36{,}864$; $PS = \$12{,}288$

28. Your video store has the exponential demand of equation $p = 15e^{-0.01q}$, where q represents daily sales of used DVD's and p represents daily price you charge per DVD. Calculate the daily Consumer's Surplus if you sell used DVDs at \$5 each.

Answer: $CS = \int_0^{\bar{q}} (D(q) - \bar{p})dq = \int_0^{109.9} (15e^{-0.01q} - 5)dq = \$450.69$

29. Calculate the Producer's Surplus for the supply equation $p = 13 + 2q$ at the unit price $\bar{p} = 29$.

Answer: $PS = \int_0^{\bar{q}} (\bar{p} - S(q))dq = \int_0^{8} (29 - (13 + 2q))dq = \$64$

30. Calculate the Producer's Surplus for the supply equation $p = 7 + 2q^{\frac{1}{3}}$ at the unit price $\bar{p} = 14$.

Answer: $\bar{q} = 3.5^3 = 42.875$; $PS = \int_0^{\bar{q}} (\bar{p} - S(q))dq = \int_0^{42.875} \left(14 - (7 + 2q^{\frac{1}{3}})\right) dq = \$75$

Improper Integral (Section 14.5)

31. Determine whether the given improper integral converges or diverges.

- If convergent, evaluate the integral and give the numerical answer.
- If divergent, indicate if the integral diverges to positive infinity or negative infinity.

a) $\int_1^{\infty} \frac{8}{x^2} dx$

Answer: $\int_1^{\infty} \frac{8}{x^2} dx = \lim_{t\to\infty} \int_1^{t} \frac{8}{x^2} dx = \lim_{t\to\infty} \left(\left(\frac{-8}{x}\right)\Big|_1^t\right) = \lim_{t\to\infty} \left(-\frac{8}{t} + 8\right) = 8$ since $\lim_{t\to\infty} \left(-\frac{8}{t}\right) = 0.$

Thus, this improper integral converges to 8.

b) $\int_{-2}^{\infty} e^{-3x}\, dx$

Answer: $\int_{-2}^{\infty} e^{-3x}\, dx = \lim_{t\to\infty} \int_{-2}^{t} e^{-3x}\, dx = \lim_{t\to\infty} -\frac{1}{3}e^{-3x}|_{-2}^{t} = \lim_{t\to\infty} \left(-\frac{1}{3}e^{-3t} - \left(-\frac{1}{3}e^{6}\right)\right) =$

$\lim_{t\to\infty} \left(-\frac{1}{3}e^{-3t} + \frac{1}{3}e^{6}\right) = \frac{1}{3}e^{6} = \frac{e^6}{3}$ since the $\lim_{t\to\infty} \left(-\frac{1}{3}e^{-3t}\right) = 0.$

Thus, this improper integral converges to $\frac{e^6}{3}$.

c) $\int_1^{\infty} \frac{3}{x} dx$

Answer: $\int_1^{\infty} \frac{3}{x} dx = \lim_{t\to\infty} \int_1^{t} \frac{3}{x} dx = \lim_{t\to\infty} 3\ln|x|\,|_1^t = \lim_{t\to\infty} (3\ln|t| - 3\ln 1) = \infty$.

Since $\lim_{t\to\infty} \ln|t| = \infty$ and $\ln 1 = 0$.

Thus, this improper integral diverges.

d) $\int_1^{\infty} \frac{6}{x^2} dx$ — **Answer**: Converges to 6

e) $\int_{-\infty}^{-2} \frac{3}{x^2} dx$ — **Answer**: Converges to $\frac{3}{2}$

f) $\int_1^{\infty} x\, dx$ — **Answer**: Diverges to ∞

g) $\int_0^{3} \frac{1}{x^{1.1}} dx$ — **Answer**: Diverges to ∞

h) $\int_0^{3} \frac{1}{x^{0.1}} dx$ — **Answer**: Converges to 2.986528

i) $\int_{-\infty}^{-1} \frac{1}{x^{\frac{1}{3}}} dx$ — **Answer**: Diverges to - ∞

j) $\int_4^{\infty} \frac{1}{x^4} dx$ — **Answer**: Converges to 0.005208

k) $\int_0^{\infty} e^{-x}\, dx$ — **Answer**: Converges to 1

l) $\int_{-\infty}^{2} e^{x}\, dx$ — **Answer**: Converges to e^2

m) $\int_4^{\infty} e^{-2x}\, dx$ — **Answer**: Converges to $\frac{1}{2e^8}$

n) $\int_3^{\infty} x^2\, dx$ — **Answer**: Diverges to ∞

o) $\int_{-\infty}^{0} e^{2x}\, dx$ — **Answer**: Converges to $\frac{1}{2}$