MAT 210 Exam Final Review Questions

The final exam is comprehensive; it will emphasize on the last part (anti-derivative). You need to review the review problems for exam 1, and 2. In addition, below are some review questions related to the sections covered after Exam 2.

**Indefinite Integral (Section 13.1)**

1. Find the indefinite integrals.
   (a) \( \int 2x^4 - 4x^{-2} + 5x^{-5} + 3 \, dx \)
   (b) \( \int \frac{7}{x} + \frac{1}{3x^7} \, dx \)
   (c) \( \int \frac{2}{x^2} - 5\sqrt{x} \, dx \)
   (d) \( \int e^x - x^{-0.3} \, dx \)
   (e) \( \int (x + 3)(x - 2) \, dx \)
   (f) \( \int \frac{x^4 + 5x - 2}{x} \, dx \)

**Answer:**

(a) \( \frac{2}{5} x^5 + \frac{4}{x} - \frac{5}{4x^4} + 3x + C \)
(b) \( 7\ln|x| - \frac{1}{18x^6} + C \)
(c) \( -\frac{2}{x} - \frac{10}{3} x^{\frac{2}{3}} + C \)
(d) \( e^x - \frac{x^{0.7}}{0.7} + C = e^x - \frac{10}{7} x^{0.7} + C \)
(e) \( \int x^2 + x - 6 \, dx = \frac{x^3}{3} + \frac{x^2}{2} - 6x + C \)
(f) \( \int x + 5 - \frac{2}{x} \, dx = \frac{x^2}{2} + 5x - 2\ln|x| + C \)

2. Find \( f(x) \) if \( f(0) = -1 \) and the derivative \( f'(x) = 9e^x + 9. \)

**Answer:** \( f(x) = 9e^x + 9x - 10 \)

3. The velocity of a particle moving in a straight line is \( v(t) = t^2 + 6. \) Find the expression for the position, \( s(t), \) of the particle at time \( t, \) if \( s(3) = 0. \)

**Answer:** \( s(t) = \frac{1}{3} t^3 + 6t - 27 \)

4. Suppose the function \( C(x) \) gives the total cost (in dollars) of producing \( x \) units of a certain product. The marginal cost of producing the \( x \)th unit is \( C'(x) = 0.5x + \frac{1}{x}. \) If the cost to produce the first unit is 5 dollars, find the cost function \( C(x). \)
Answer: \( C(x) = 0.25x^2 + \ln |x| + 4.75 \) dollars

\[
C(x) = \int C'(x) \, dx = \int 0.5x + \frac{1}{x} \, dx = 0.25x^2 + \ln |x| + K
\]
\( C(1) = 5 \), So \( 5 = 0.25 \cdot 1^2 + \ln |1| + K \). Then solve for constant \( K \): \( K = 5 - 0.25 = 4.75 \).

Substitution (Section 13.2)

5. Use integration by substitution to find the integrals.

(a) \( \int 16e^{-3x} \, dx \) (can also use short-cut formula)
(b) \( \int (5x - 2)^3 \, dx \) (can also use short-cut formula)
(c) \( \int \frac{1}{2x-5} \, dx \) (can also use short-cut formula)
(d) \( \int 4xe^{x^2-3} \, dx \)
(e) \( \int x(x^2 + 1)^{10} \, dx \)
(f) \( \int 15x\sqrt{x^2 + 7} \, dx \)
(g) \( \int (3x^2 + 1)(x^3 + x - 2)^9 \, dx \)

Answer:

(a) \( 16 \cdot e^{-3x} \cdot \frac{1}{3} + C = \frac{16}{3} e^{-3x} + C \)
(b) \( \frac{(5x-2)^4}{4} \cdot \frac{1}{5} + C = \frac{1}{20} (5x - 2)^4 + C \)
(c) \( \frac{1}{2} \ln|2x - 5| + C \)
(d) \( 2e^{x^2-3} + C \)
(e) \( \frac{1}{22} (x^2 + 1)^{11} + C \)
(f) \( -5(-x^2 + 7)^\frac{3}{2} + C \)
(g) \( \frac{1}{10} (x^3 + x - 2)^{10} + C \)

Fundamental Theorem of Calculus; Definite Integral; Left Riemann Sum (Sections 13.3, 13.4)

6. Evaluate the definite integrals.

(a) \( \int_0^1 (6x^5 + 15x^4 - 9x^2 + 1) \, dx \)
(b) \( \int_2^7 \left( x + \frac{5}{x} \right) \, dx \)
(c) \( \int_1^{10} \frac{1}{x^2} \, dx \)
(d) \( \int_0^6 e^{-x+6} \, dx \)
(e) \( \int_{-1}^1 5e^{3x} \, dx \)
(f) \( \int_{e^5}^{e^6} \frac{2}{x} \, dx \)
(g) \( \int_{\ln 3}^{\ln 5} e^{2x} \, dx \)

**Answer:**

(a) 2
(b) \( \frac{45}{2} + 5 \ln \left( \frac{7}{2} \right) \)
(c) \( \frac{9}{10} \)
(d) \(-1 + e^6\)
(e) \( \frac{5}{3} (e^3 - e^{-3}) \)
(f) 4
(g) 8

7. Assume that \( b \) is a positive number, solve the following equation for \( b \).

\[ \int_{2}^{b} (2x - 4) \, dx = 9 \]

**Answer:** \( b = 5 \)

8. Calculate the left Riemann sum for the function \( f(x) = 3x^2 + 2x - 3 \) over the interval \([1, 3]\), with \( n = 5 \).

**Answer:** 22.56

\[ \Delta x = \frac{b-a}{n} = \frac{5-1}{5} = 0.4, \quad x_0 = a = 1, \quad x_1 = x_0 + \Delta x = 1.4, \quad x_2 = 1.8, \quad x_3 = 2.2, \quad x_4 = 2.6. \]

\[ LRS = \Delta x \cdot (f(1) + f(1.4) + f(1.8) + f(2.2) + f(2.6)) = 0.4(2 + 5.68 + 10.32 + 15.92 + 22.48) = 22.56 \]

9. Use a left Riemann sum to estimate the definite integral with \( n = 4 \) subintervals.

\[ \int_{2}^{3} \frac{1}{1 + 2x} \, dx \]

**Answer:** 0.18

\[ \Delta x = 0.25, \quad LRS = 0.25 \left( \frac{1}{1+2(2)} + \frac{1}{1+2(2.25)} + \frac{1}{1+2(2.5)} + \frac{1}{1+2(2.75)} \right) = 0.25(0.2 + 0.18 + 0.17 + 0.15) = 0.18 \]

**Applications of Definite Integrals (Section 13.4)**

10. A particle moves in a straight line with velocity \( v(t) = -t^2 + 8 \) meters per second, where \( t \) is time in seconds.

Find the displacement of the particle between \( t = 2 \) and \( t = 6 \) seconds.

**Answer:** \(-37\) meters

\[ \text{Displacement} = s(6) - s(2) = \int_{2}^{6} v(t) \, dt = \int_{2}^{6} (-t^2 + 8) \, dt = -\frac{112}{3} \approx -37 \text{ meters}. \]
11. The marginal revenue of the $x$th box of flash cards sold is $500e^{-0.001x}$ dollars. Find the revenue generated by selling box 101 through 5,000.

**Answer:** 448,598 dollars

Total revenue generated $= R(5000) - R(101) = \int_{101}^{5000} MR \, dx = \int_{101}^{5000} 500e^{-0.001x} \, dx \approx 448597.54$ dollars

12. Since YouTube first became available to the public in mid-2005, the rate at which video has been uploaded to this site can be approximated by $f(t) = 1.1t^2 - 2.6t + 2.3$ million hours of videos per year ($0 \leq t \leq 9$), where $t$ is time in years since June 2005. Use a definite integral to estimate the total number of hours of video uploaded from June 2007 to June 2010.

**Answer:** 23 million hours of video

Total number of hours $= \int_2^5 f(t) \, dt = \int_2^5 (1.1t^2 - 2.6t + 2.3) \, dt \approx 23$ million hours of video

13. Calculate the area of the region bounded by $y = \sqrt{x}$, the $x$-axis, and the lines $x = 0$ and $x = 16$.

**Answer:** $\frac{128}{3}$

Area under curve $= \int_0^{16} \sqrt{x} \, dx = \frac{128}{3}$

**Integration by Parts (Section 14.1)**

Integration by parts formula: $\int u \, dv = uv - \int v \, du$

14. Use integration by parts to find the integrals.

(a) $\int 2xe^xdx$

(b) $\int (3x + 4)e^{-5x}dx$

(c) $\int \ln x \, dx$

(d) $\int x^2 \ln x \, dx$

**Answer:**

(a) Let $u = 2x$, $dv = e^x \, dx$. Then $du = 2 \, dx$ and $v = e^x$. Using the formula $\int u \, dv = uv - \int v \, du$ to get

$\int 2xe^x \, dx = 2xe^x - \int e^x \, 2 \, dx = 2xe^x - 2e^x + C$

(b) $-\frac{1}{5}(3x + 4)e^{-5x} - \frac{3}{25}e^{-5x} + C = \left(-\frac{3}{5}x - \frac{23}{25}\right)e^{-5x} + C$

(Let $u = 3x + 4$, $dv = e^{-5x} \, dx$. Then $du = 3 \, dx$ and $v = -\frac{1}{5}e^{-5x}$)

(c) $x \ln x - x + C$ (Let $u = \ln x$, $dv = dx$)

(d) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$ (Let $u = \ln x$, $dv = x^2 \, dx$)
Area between Curves (Section 14.2)

15. Find the area of the region enclosed by the curves of $y = -x^2 + 6x + 2$ and $y = 2x^2 + 9x - 4$.

**Answer:** 13.5

Find the intersection points: $-x^2 + 6x + 2 = 2x^2 + 9x - 4$

$0 = 3x^2 + 3x - 6$

$0 = 3(x + 2)(x - 1)$

So $x = -2$ and $x = 1$.

The area enclosed by the curves from $-2$ to $1$ is

$$\int_{-2}^{1} (\text{top} - \text{bottom}) \, dx = \int_{-2}^{1} [(-x^2 + 6x + 2) - (2x^2 + 9x - 4)] \, dx$$

$$= \int_{-2}^{1} (-3x^2 - 3x + 6) \, dx$$

$$= -x^3 - \frac{3}{2}x^2 + 6x \bigg|_{-2}^{1}$$

$$= (-1^3 - \frac{3}{2}1^2 + 6(1)) - \left((-2)^3 - \frac{3}{2}(-2)^2 + 6(-2)\right) = 13.5$$

16. Find the area of the region enclosed by the curves of $f(x) = x^2 - x + 5$ and $g(x) = x + 8$.

**Answer:** $\frac{32}{3}$

17. Find the area of the region between $y = x^2$ and $y = -1$ from $x = -1$ and $x = 1$.

**Answer:** $\frac{8}{3}$

18. Which of the following calculates the area of the region(s) between the curves $y = x^2$ and $y = 1$ from $x = -1$ to $x = 2$?

A. $\int_{-1}^{2} (x^2 - 1) \, dx$

B. $\int_{-1}^{2} (1 - x^2) \, dx$

C. $\int_{-1}^{1} (1 - x^2) \, dx + \int_{1}^{2} (x^2 - 1) \, dx$

D. $\int_{-1}^{1} (x^2 - 1) \, dx + \int_{1}^{2} (1 - x^2) \, dx$

E. None of the above.

**Answer:** C
19. Find the average value of \( f(x) = 6e^{0.5x} \) over the interval \([-1, 3]\).

**Answer:** \( 3(e^{1.5} - e^{-0.5}) \)

The average value of a continuous function \( f(x) \) over interval \([a, b]\) is 

\[
\frac{1}{b-a} \int_a^b f(x) \, dx.
\]

\[
\frac{1}{3 - (-1)} \int_{-1}^3 6e^{0.5x} \, dx = \frac{1}{4} \cdot 6 \cdot e^{0.5x}_{\mid_1} = 3e^{0.5x}_{\mid_1} = 3(e^{1.5} - e^{-0.5})
\]

20. Find the average of the function \( f(x) = x^3 - x \) over the interval \([0, 2]\).

**Answer:** 1

21. Find the average value of the function \( f(x) = 6x^2 - 4x + 7 \) over the interval \([-2, 2]\).

**Answer:** 15

**Consumer’s Surplus and Producer’s Surplus (section 14.4)**

22. Your shop can sell 50 “I love calculus” t-shirt at $20 each per day. You decided to drop the price $1.50 per shirt and this results in 3 more t-shirts sold per day.

a. Write a linear function for the unit price \( p \) of the t-shirt sold daily as a function of \( q \) (demand), the number t-shirts can be sold at unit price \( p \).

**Answer:** When \( p = 20 \), the demand \( q = 50 \) and when the unit price \( p = 18.5 \), the demand \( q = 53 \). The slope is \( \frac{-1.5}{5} = -\frac{1}{2} \) dollar per t-shirt. Thus, \( p - 20 = -\frac{1}{2}(q - 50) \), that is \( p = -\frac{1}{2}q + 45 \).

b. Calculate the consumers’ surplus when the unit price is \( \bar{p} = 15 \) dollars per shirt using the demand equation found in part (a). Consumers’ Surplus is defined as \( \int_0^{\bar{q}} (D(q) - \bar{p})dq \).

**Answer:** When \( \bar{p} = 15 \), then \( 15 = -\frac{1}{2}q + 45 \), \(-30 = -\frac{1}{2}q \), \( \bar{q} = 60 \).

\[
CS = \int_0^{60} ((-\frac{1}{2}q + 45) - 15)dq = \int_0^{60} \left(-\frac{1}{2}q + 30\right)dq = \left(-\frac{1}{4}q^2 + 30q\right)_0^{60} = -\frac{1}{4} \cdot 60^2 + 30 \cdot 60 = -900 + 1800 = $900.
\]

The consumers’ surplus is 900 dollars for the first 60 t-shirt sold.

23. Calculate the producers’ surplus for the supply equation at the indicated unit price \( \bar{p} = 160 \) (Round your answer to the nearest cent.)

Producers’ Surplus is defined a \( \int_0^{\bar{q}} (\bar{p} - S(q))dq \) where \( p = 130 + e^{0.01q}, \bar{p} = 160 \).

**Answer:** When \( \bar{p} = 160 \), then \( 160 = 130 + e^{0.01q}, 30 = e^{0.01q}, \bar{q} = \frac{\ln 30}{0.01} = 340.1197382 \).

\[
PS = \int_0^{\frac{\ln 30}{0.01}} (160 - (130 + e^{0.01q})) = \int_0^{\frac{\ln 30}{0.01}} (30 - e^{0.01q})dq = \left(30q - \frac{1}{0.01}e^{0.01q}\right)_{\frac{\ln 30}{0.01}}^{\frac{\ln 30}{0.01}}
\]
The producers’ surplus is $ 7303.59 for the first 340 items sold.

24. A company finds that the demand for their new product is given by \( p = 13 - q^\frac{1}{2} \), where \( p \) is the price per item and \( q \) is the number of items that can be sold per week at unit price \( p \). The company is prepared to sell \( q = \left( \frac{p-4}{2} \right)^4 \) items per week at a unit price \( p \). Find the equilibrium price \( \bar{p} \) and the consumers’ and producers’ surpluses at the equilibrium price. What is the total social gain at the equilibrium price?

**Answer:** To find the equilibrium price solve the following system of equations \( p = 13 - q^\frac{1}{2} \) and \( q = \left( \frac{p-4}{2} \right)^4 \) simultaneously, that is find when the supply meets the demand. Substitute \( q = \left( \frac{p-4}{2} \right)^4 \) for \( q \) into \( p = 13 - q^\frac{1}{2} \).

Using the identity \((a^k)^n = a^{kn}\), we obtain \( p = 13 - q^\frac{1}{2} = 13 - \left( \frac{p-4}{2} \right)^4 \times \frac{1}{2} = 13 - \left( \frac{p-4}{2} \right)^2 \). Solve the equation \( p = 13 - \left( \frac{p-4}{2} \right)^2 \) for \( p \). Multiplying by 2, we get \( 2p = 26 - (p - 4) = 30 - p \). Then \( 3p = 30 \) and \( \bar{p} = $10 \).

To find the corresponding demand \( \bar{q} \), substitute \( \bar{p} = 10 \) into \( q = \left( \frac{p-4}{2} \right)^4 \) and solve for \( q \). Thus \( \bar{q} = 81 \).

The equilibrium price is \( \bar{p} = 10 \).

The consumers’ surplus for the first 81 items is \( \int_{0}^{81} (13 - \frac{1}{2} q^{1/2} - 10) dq = \left( 3q - \frac{1}{125} q^{1.25} \right)|_{0}^{81} = 3 \cdot 81 - \frac{1}{125} \cdot 81^{1.25} = $48.6 \)

The producers’ surplus for the first 81 items is 97.2 dollars.

The total social gain is the sum of CS and PS, that is \( 48.6 + 97.2 = 145.8 \) dollars for the first 81 items.

25. Given that \( x \) is number of items, the demand function is \( d(x) = 200 - 0.2x \), and the supply function is \( s(x) = 0.3x \).

a) Find the equilibrium quantity.

b) Find the consumers’ surplus and producers’ surplus at the equilibrium quantity.

**Answer:** a) \( x = 400 \) items

b) \( CS = $16,000 \); \( PS = $24,000 \)

26. Given that \( x \) is number of items, the demand function is \( d(x) = 270.4 - 0.1x^2 \), and the supply function is \( s(x) = 0.3x^2 \).
a) Find the equilibrium quantity.
b) Find the consumers’ surplus and producers’ surplus at the equilibrium quantity.

**Answer:**

a) \( x = 26 \) items  
b) \( CS = \$1,171.73 \); \( PS = \$3,515.2 \)

27. Given that \( x \) is number of items, the demand function is \( d(x) = \frac{2304}{\sqrt{x}} \), and the supply function is \( s(x) = 9\sqrt{x} \).

a) Find the equilibrium quantity.
b) Find the consumers’ surplus and producers’ surplus at the equilibrium quantity.

**Answer:**

a) \( x = 256 \) items  
b) \( CS = \$36,864 \); \( PS = \$12,288 \)

28. Your video store has the exponential demand of equation \( p = 15e^{-0.01q} \), where \( q \) represents daily sales of used DVD’s and \( p \) represents daily price you charge per DVD. Calculate the daily Consumer’s Surplus if you sell used DVDs at $5 each.

**Answer:** \( CS = \int_0^q (D(q) - \bar{p})dq = \int_0^{109.9} (15e^{-0.01q} - 5)dq = \$450.69 \)

29. Calculate the Producer’s Surplus for the supply equation \( p = 13 + 2q \) at the unit price \( \bar{p} = 29 \).

**Answer:** \( PS = \int_0^q (\bar{p} - S(q))dq = \int_0^8 (29 - (13 + 2q))dq = \$64 \)

30. Calculate the Producer’s Surplus for the supply equation \( p = 7 + 2q^{\frac{1}{3}} \) at the unit price \( \bar{p} = 14 \).

**Answer:** \( \bar{q} = 3.5^3 = 42.875 \); \( PS = \int_0^q (\bar{p} - S(q))dq = \int_0^{42.875} \left(14 - \left(7 + 2q^{\frac{1}{3}}\right)\right) dq = \$75 \)

**Improper Integral (Section 14.5)**

31. Determine whether the given improper integral converges or diverges.

- If convergent, evaluate the integral and give the numerical answer.
- If divergent, indicate if the integral diverges to positive infinity or negative infinity.

a) \( \int_1^\infty \frac{8}{x^2} dx \)

**Answer:** \( \int_1^\infty \frac{8}{x^2} dx = \lim_{t \to \infty} \int_1^t \frac{8}{x^2} dx = \lim_{t \to \infty} \left(-\frac{8}{x}\right)_1^t = \lim_{t \to \infty} \left(-\frac{8}{t} + 8\right) = 8 \) since \( \lim_{t \to \infty} \left(-\frac{8}{t}\right) = 0 \).

Thus, this improper integral converges to 8.

b) \( \int_{-2}^\infty e^{-3x} dx \)

**Answer:** \( \int_{-2}^\infty e^{-3x} dx = \lim_{t \to \infty} \int_{-2}^t e^{-3x} dx = \lim_{t \to \infty} \left[-\frac{1}{3}e^{-3x}\right]_{-2}^t = \lim_{t \to \infty} \left(-\frac{1}{3}e^{-3t} - (-\frac{1}{3}e^6)\right) = \lim_{t \to \infty} \left(-\frac{1}{3}e^{-3t} + \frac{1}{3}e^6\right) = e^6 \) since the \( \lim_{t \to \infty} \left(-\frac{1}{3}e^{-3t}\right) = 0 \).

Thus, this improper integral converges to \( e^6 \).
c) \[ \int_1^\infty \frac{3}{x} \, dx \]

**Answer:** \[ \int_1^\infty \frac{3}{x} \, dx = \lim_{t \to \infty} \int_1^t \frac{3}{x} \, dx = \lim_{t \to \infty} 3 \ln |x| \bigg|_1^t = \lim_{t \to \infty} (3 \ln |t| - 3 \ln 1) = \infty. \]

Since \( \lim_{t \to \infty} \ln |t| = \infty \) and \( \ln 1 = 0 \).

Thus, this improper integral diverges.

d) \[ \int_1^\infty \frac{6}{x^2} \, dx \]

**Answer:** Converges to 6

e) \[ \int_{-\infty}^{-2} \frac{3}{x^2} \, dx \]

**Answer:** Converges to \( \frac{3}{2} \)
f) \[ \int_1^\infty x \, dx \]

**Answer:** Diverges to \( \infty \)
g) \[ \int_0^3 \frac{1}{x^{1.1}} \, dx \]

**Answer:** Diverges to \( \infty \)
h) \[ \int_0^3 \frac{1}{x^{0.1}} \, dx \]

**Answer:** Converges to 2.986528

i) \[ \int_{-\infty}^{-1} \frac{1}{x^3} \, dx \]

**Answer:** Diverges to \( -\infty \)

j) \[ \int_4^\infty \frac{1}{x^7} \, dx \]

**Answer:** Converges to 0.005208

k) \[ \int_0^\infty e^{-x} \, dx \]

**Answer:** Converges to 1

l) \[ \int_{-\infty}^{0} e^x \, dx \]

**Answer:** Converges to \( e^2 \)

m) \[ \int_4^\infty e^{-2x} \, dx \]

**Answer:** Converges to \( \frac{1}{2e^8} \)

n) \[ \int_3^\infty x^2 \, dx \]

**Answer:** Diverges to \( \infty \)

o) \[ \int_{-\infty}^{0} e^{2x} \, dx \]

**Answer:** Converges to \( \frac{1}{2} \)