MAT 210 Final Exam Review

Limits and Continuity

1. Calculate the limit: $\lim_{x \to \infty} \frac{5-8x^3}{7-9x-4x^3}$ Answer: 2

2. Calculate the limit: $\lim_{x \to \infty} \frac{3-2x^2}{2-4x^2+x^5}$ Answer: 0

3. Calculate the limit: $\lim_{x \to 2} \frac{x^2-5x+6}{x^2-2x}$ Answer: -1/2

4. Calculate the limit: $\lim_{x \to 3} \frac{x^2-9}{2x-6}$ Answer: 3

5. Calculate the limit: $\lim_{x \to -7} \frac{x^2+3x-2}{x^2+8x+7}$ Answer: 11/6

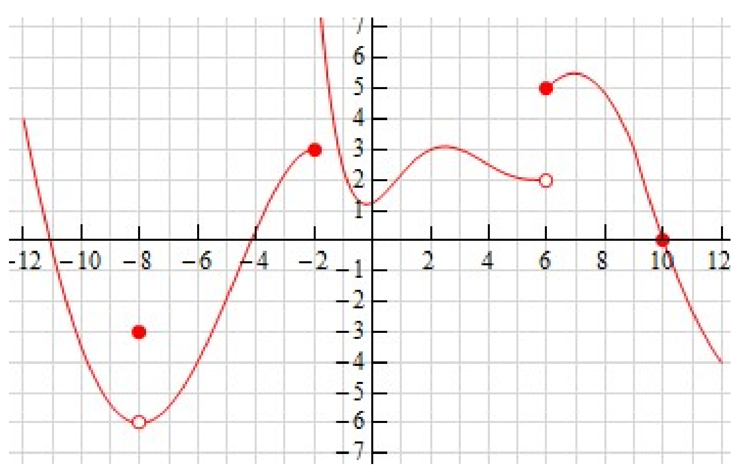
6. Calculate the limit: $\lim_{x \to 4^-} \frac{7x-1}{x-4}$ Answer: $-\infty$

7. Calculate the limit: $\lim_{x \to 4^+} \frac{7x-1}{x-4}$ Answer: ∞

8. Let $f(x) = \begin{cases} 10, & x < 4 \\ 3x+1, & x \geq 4 \end{cases}$

   Is f continuous at $x = 4$? Justify your answer. Answer: No

9. Use the graph of function $f(x)$ answer the questions. Write DNE if a limit does not exist.



(a) $\lim_{x\to6^+} f(x)$ Answer: 5

(b) $\lim_{x\to6^-} f(x)$ Answer: 2

(c) $\lim_{x\to6} f(x)$ Answer: DNE

(d) $f(6)$ Answer: 5

(e) $\lim_{x\to-8} f(x)$ Answer: -6

(f) $f(-8)$ Answer: - 3

(g) $\lim_{x\to-2} f(x)$ Answer: DNE

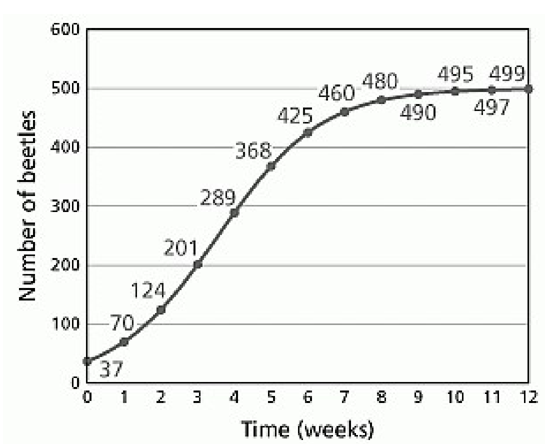
(h) $\lim_{x\to10} f(x)$ Answer: 0

(i) List the x-values of discontinuous point(s) of the function $f(x)$. Answer: -8, -2, 6

Rates of Change

10. Let $f(x) = x^3 + 2$. Find the average rate of change of f over the interval $[1, 4]$.
Answer: 21

11. The graph below shows the population of beetles in a greenhouse t weeks after the season's flowers were planted.



a) Calculate the average rate of change over the interval [1,10].
Answer: 425/9

b) Circle **T** for True or **F** for False next to each statement.

T F a) During weeks [4,12] the instantaneous rate of change of the population is increasing

T F b) The average rate of change of the population on [4,12] is less than the instantaneous rate of change of the population at $t = 4$

T F c) The average rate of change of the population on [0,12] is greater than the instantaneous rate of change of the population at $t = 2$

T F d) The instantaneous rate of change of the population first increased then decreased

Differentiation (Power Rule, Product Rule, Quotient Rule, Contant Multiple Rule)

12. Find the derivative of each function using differentiation rules.

a) $f(x) = \frac{8}{\sqrt{x}} - \frac{4}{x}$ Answer: $f'(x) = -4x^{-3/2} + 4x^{-2}$

b) $g(x) = 3x^2 - \frac{2}{x^6} - 5x^{2.3} + 7.95$ Answer: $g'(x) = 6x + 12x^{-7} - 11.5x^{1.3}$

c) $m(x) = -5x^2e^{-4x+3}$ Answer: $m'(x) = -10xe^{-4x+3} + 20x^2e^{-4x+3}$

d) $v(x) = \frac{2x-1}{4x+8}$ Answer: $v'(x) = \frac{20}{(4x+8)^2}$

Tangent Lines

13. Find the equation of the tangent line to the graph of $f(x) = 3x^2 - 4x + 6$ at $x = 2$.

Answer: $y = 8x - 6$

Marginal Analysis

14. Assume that your monthly profit (in dollars) from selling homemade cookies is given by $P(x) = 8x - 2\sqrt{x}$, where x is the number of boxes of cookies you sell in a month.

a) Determine the marginal profit function, *MP(x)*.

Answer: $MP(x) = 8 - \frac{1}{\sqrt{x}}$

b) Determine value of marginal profit if you are selling 25 boxes of cookies per month and interpret.

Answer: 7.8 dollars per box. After 25 boxes of cookies have been sold, the total profit will increase by about 7.8 dollars per additional box sold, or the profit from selling the 26th box is about 7.8 dollars.

15. Your monthly cost (in dollars) from selling homemade candles is given by $C(x) = 150 + 0.1x + 0.002x^2$, where *x* is the number of candles you sell in a month. The revenue from selling x candles is $R(x) = 7x$.

a) Write a function $P(x)$ for your monthly *profit* of producing and selling *x* candles.
Answer: $P(x) = -150 + 6.9x - 0.002x^2$

b) Calculate $P(100)$. Include units.
Answer: 520 dollars

c) Write a function for your *marginal profit*.
Answer: $MP(x) = 6.9 - 0.004x$

d) Calculate your marginal profit if you produce and sell 100 candles. Include units and interpret your answer.
Answer: 6.5 dollars per candle. The profit from selling the 101st candle is about 6.5 dollars. Or the total profit will increase by 6.5 dollars per candle sold, after 100 candles are sold.

Average Velocity and Instantaneous Velocity

16. Assume that the distance, s (in meters), traveled by a car moving in a straight line is given by the function $s(t) = t^2 - 3t + 5$, where t is measured in seconds.

a) Find the *average* velocity of the car during the time period from $t = 1$ to $t = 4$.

Answer: 2 m/s

b) Find the *instantaneous* velocity of the car at time $t = 3$ seconds.

Answer: 3 m/s

Chain Rule

17. Find the derivative y' of each function.

(a) $y = 0.4(3x^2 + 2x - 8)^5$
(b) $y = 3\ln(5x^4 + 4x)$
(c) $y = 8e^{-2x}$
(d) $y = (x^2 - 2x) \cdot e^{2x+3}$
(e) $y = \sqrt{x^3 - 50x}$

Answer:

(a) $2(3x^2 + 2x - 8)^4(6x + 2)$

(b) $\frac{3(20x^3+4)}{5x^4+4x}$

(c) $-16e^{-2x}$

(d) $(2x - 2)e^{2x+3} + (x^2 - 2x)e^{2x+3} \cdot 2 = (2x^2 - 2x - 2)e^{2x+3}$

(e) $\frac{1}{2}(x^3 - 50x)^{-\frac{1}{2}} \cdot (3x^2 - 50)$

Implicit Differentiation

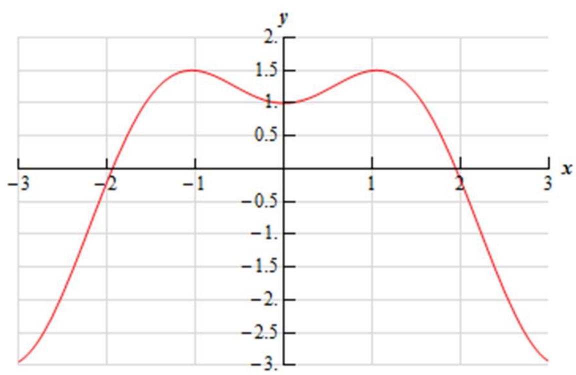
18. Find the derivative $\frac{dy}{dx}$.
    (a) $x^3 - y^3 + y = 3$
    (b) $6x^2y - 15x = y^2$

Answer:

(a) $\frac{dy}{dx} = \frac{3x^2}{3y^2-1}$
(b) $\frac{dy}{dx} = \frac{15-12y}{6x^2-2y}$

Absolute and Relative Maxima and Minima

19. The function of a function f on $[-3, 3]$ is given below.
    (a) f has a relative minimum at $x =$
    (b) f has an absolute maximum at $x =$
    (c) f has an absolute minimum at $x =$
    (d) f' is zero at $x =$
    (e) f' is positive on interval(s):
    (f) f' is negative on interval(s):



Answer:

(a) 0
(b) $-1, 1$
(c) $-3, 3$
(d) $-1, 0, 1$
(e) $(-3, -1)$ and $(0, 1)$
(f) $(-1, 0)$ and $(1, 3)$

20. Consider the function $f(x) = 8x^3 - 24x + 12$.
    (a) Find all critical points of f.
    (b) Find the absolute extrema of f on interval $[-3, 2]$.

    Answer:
    (a) Critical points: $x = -1, 1$.
    (b) Abs Max = $f(-1) = 28$; Abs Min= $f(-3) = -132$.

21. For the function $k(x) = 4x^3 - 24x^2 + 36x - 20$,
    (a) Find any critical points of k.
    (b) For what x values is the function k increasing? decreasing?
    (c) Find any relative and absolute extrema of k on $[-1, 5]$.

    Answer:

    (a) Critical points: $x = 1, 3$.
    (b) Increasing on $(-1, 1) \cup (3, 5)$; decreasing on $[1, 3]$.
    (c) Abs Max = $k(5) = 60$; Abs Min = $k(-1) = -84$; Rel Max = $k(1) = -4$; Rel Min = $k(3) = -20$.

Optimization: Applications to Maximum and Minimum

22. You are running a business selling homemade bread. Your weekly revenue from the sale of q loaves bread is $R(q) = 68q - 0.1q^2$ dollars, and the weekly cost of making q loaves of bread is $C(q) = 23 + 20q$.
    (a) Find the weekly profit function $P(q)$.
    (b) Find the production level q that maximizes the weekly profit.
    (c) Find the maximum profit.

    Answer:

    (a) $P(q) = 68q - 0.1q^2 - (23 + 20q) = -0.1q^2 + 48q - 23$
    (b) Set $P'(q) = -0.02q + 48 = 0$, and solve for q: $q = 240$
    (c) $P(240) = -0.1 \cdot 240^2 + 48 \cdot 240 - 23 = 5737$ dollars.

23. Worldwide annual sale of a product in 2013-2017 were projected to be approximately $q = -10p + 4220$ million units at a selling price of p dollars per unit. What *selling price* would have resulted in the largest projected annual revenue? What would be resulting revenue?

    Answer: $R(p) = p \cdot q = p(-10p + 4220) = -10p^2 + 4220p$. Set $R'(p) = -20p + 4220 = 0$ and solve for p: $p = \$210$. Maximum revenue is $R(210) = 210 \cdot 2120 = \$445200$

24. Suppose $C(x) = 0.02x^2 + 2x + 4000$ is the total cost for a company to produce x units of a certain product. Find the production level x that minimizes the average cost $\bar{C}(x) = \frac{C(x)}{x}$.

Answer: $\bar{C}(x) = 0.02x + 2 + 4000x^{-1}$. Set $\bar{C}'(x) = 0.02 - 4000x^{-2} = 0$ and solve for x: $x = 447$.

25. I need to create a rectangular vegetable patch with an area of exactly 162 square feet. The fencing for the east and west sides costs $4 per foot, and the fencing for the north and south sides costs only $2 per foot. What are the dimensions of the vegetable patch with the *least expensive* fence?

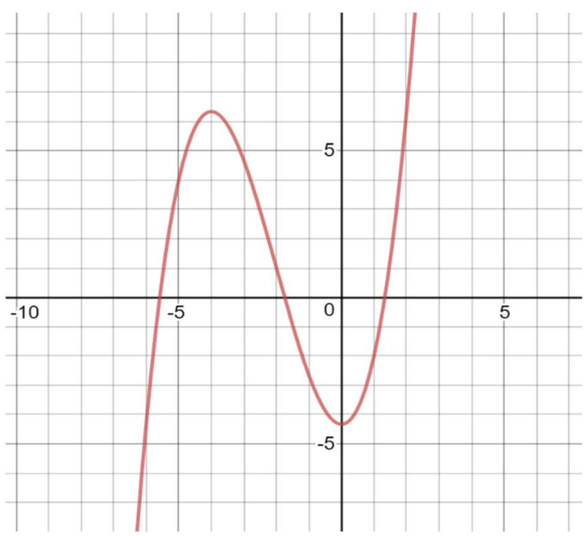
Answer:

Minimize the cost $C = 4x + 8y$ subject to area $xy = 162$, where x is the length of the north and south sides, and y is the length of east and west sides.

Dimension is $x = 18, y = 9$. Minimum cost $= 144$ dollars.

Higher-order Derivatives, Acceleration and Concavity

26. The graph of a function $y = f(x)$ is given below.
   (a) $f'(-4) = f'(0) =$
   (b) Is $f''(-4)$ is positive or negative? Is $f''(0)$ is positive or negative?
   (c) If f has a point of inflection at $x = -2$, then $f''(-2) =$
   (d) f is concave________(up/down) on interval $(-\infty, -2)$, and concave________(up/down) on interval $(-2, \infty)$.



Answer: (a) 0; (b) negative, positive; (c) 0; (d) down, up.

27. Suppose the position of a particle moving on a straight line is $s(t) = \sqrt{t} + 4t^2$. Find the particle's acceleration as a function of time t.

Answer: $a(t) = s''(t) = -\frac{1}{4}t^{-\frac{3}{2}} + 8$

Related Rates

28. The radius of a circular puddle is growing at a rate of 15 cm/sec. How fast is its area growing at the instant when the radius is 30 cm?

    Answer: $2827 \text{ cm}^2/\text{sec}$

29. A rather flimsy spherical balloon is designed to pop at the instant its radius has reached 6 cm. Assuming the balloon is filled with helium at a rate of 13 cubic centimeters per second, calculate how fast the radius is growing at the instant it pops.

    Answer: 0.03 cm/sec

Elasticity

30. The weekly sales of some backpacks is given by $q = 1080 - 18p$, where the q represents the quantity of backpacks sold at price p.
    (a) Find the elasticity of demand at the price of \$20. Interpret your answer.
    (b) Is the demand at the price \$20 elastic, inelastic, or unit elastic? Should the price be raised or lowered from \$20 to increase the revenue?
    (c) What price will maximize the revenue?
    (d) What is the maximum weekly revenue?

    Answer:
    (a) $E(p) = -(-18) \cdot \frac{p}{1080-18p} = \frac{18}{1080-18p}$, so $E(20) = \frac{18 \cdot 20}{1080-18 \cdot 20} = 0.5$.
    This means the demand will drop by 0.5% for 1% increase from current price \$20.
    (b) $0.5 < 1$, it is inelastic. The price should be raised to increase revenue.
    (c) Solve for the price when $E(p) = 1$. Solving $\frac{18p}{1080-18p} = 1$ gives $p = \$30$.
    (d) $R = pq = 30(1080 - 18 \cdot 30) = 16200$ dollars.

31. Suppose the demand function is $q = -2p^2 + 33p$, where q represents the quantity sold at price p.
    (a) Find the price elasticity of demand $E(p)$.
    (b) Find the elasticity when $p = \$15$. If the price increase by 1%, the demand will drop by how much? Should the price be lowered or raised from \$15 to increase the revenue?

    Answer:
    (a) $E(p) = -(-4p + 33) \cdot \frac{p}{-2p^2+33p} = \frac{4p-33}{-2p+33}$
    (b) $E(15) = \frac{4(15)-33}{-2(15)+33} = \frac{27}{3} = 9 > 1$. It is elastic. The demand will drop by 9% if the price increases by 1%.
    The price should be lowered from \$15 to increase revenue.

Indefinite Integral

32. Find the indefinite integrals.
    (a) $\int 2x^4 - 4x^{-2} + 5x^{-5} + 3\ dx$
    (b) $\int \frac{7}{x} + \frac{1}{3x^7} dx$
    (c) $\int 5\sqrt{x} + e^x\, dx$

    ***Answer*:**

    (a) $\frac{2}{5}x^5 + \frac{4}{x} - \frac{5}{4x^4} + 3x + C$

    (b) $7\ln|x| - \frac{1}{18x^6} + C$

    (c) $\frac{10}{3}x^{\frac{3}{2}} + e^x + C$

33. Suppose the function $C(x)$ gives the total cost (in dollars) of producing x units of a certain product. The marginal cost of producing the xth unit is $C'(x) = 0.5x + \frac{1}{x}$. If the cost to produce the first unit is 5 dollars, find the cost function $C(x)$.

    Answer: $C(x) = 0.25x^2 + \ln|x| + 4.75$ dollars

$$C(x) = \int C'(x)\,dx = \int 0.5x + \frac{1}{x} dx = 0.25x^2 + \ln|x| + K$$

$C(1) = 5$, So $5 = 0.25 \cdot 1^2 + \ln|1| + K$. Then solve for constant K: $K = 5 - 0.25 = 4.75$.

Substitution

34. Use integration by substitution to find the integrals.
    (a) $\int 16e^{-3x}\, dx$ (can also use short-cut formula)
    (b) $\int 10(5x-2)^3 dx$ (can also use short-cut formula)
    (c) $\int \frac{1}{2x-5} dx$ (can also use short-cut formula)
    (d) $\int 4xe^{x^2-3} dx$

    ***Answer*:**

    (a) $16 \cdot \frac{e^{-3x}}{-3} + C = -\frac{16}{3}e^{-3x} + C$
    (b) $10 \cdot \frac{(5x-2)^4}{4} \cdot \frac{1}{5} + C = \frac{1}{2}(5x-2)^4 + C$
    (c) $\frac{1}{2}\ln|2x-5| + C$
    (d) $2e^{x^2-3} + C$

Definite Integral; Left Riemann Sum

35. Evaluate the definite integrals.
    (a) $\int_0^1 (6x^5 + 15x^4 - 9x^2 + 1)\, dx$
    (b) $\int_2^7 \left(x + \frac{5}{x}\right) dx$
    (c) $\int_1^{10} \frac{1}{x^2} dx$
    (d) $\int_0^2 9e^{-3x}\, dx$

Answer:

(a) 2
(b) $\frac{45}{2} + 5\ln\left(\frac{7}{2}\right)$
(c) $\frac{9}{10}$
(d) $3(1 - e^{-6})$

36. Use a Left Riemann sum to estimate the definite integral with $n = 4$ subintervals.

$$\int_2^3 \frac{1}{1 + 2x} dx$$

Answer: 0.18

$\Delta x = 0.25$, LRS $= 0.25\left(\frac{1}{1+2(2)} + \frac{1}{1+2(2.25)} + \frac{1}{1+2(2.5)} + \frac{1}{1+2(2.75)}\right) = 0.25(0.2 + 0.18 + 0.17 + 0.15) = 0.18$

Fundamental Theorem of Calculus; Applications of Definite Integrals

37. A particle moves in a straight line with velocity $v(t) = -t^2 + 8$ meters per second, where t is time in seconds. Find the displacement of the particle between $t = 2$ and $t = 6$ seconds.

Answer: -37 meters

Displacement $= s(6) - s(2) = \int_2^6 v(t)\, dt = \int_2^6 -t^2 + 8\, dt = -\frac{112}{3} \approx -37$ meters.

38. The marginal revenue of the xth box of flash cards sold is $500e^{-0.001x}$ dollars. Find the revenue generated by selling box 101 through 5,000.

Answer: 448,598 dollars

Total revenue generated $= R(5000) - R(101) = \int_{101}^{5000} MR\,dx = \int_{101}^{5000} 500e^{-0.001x}\,dx \approx 448597.54$ dollars

39. Since YouTube first became available to the public in mid-2005, the rate at which video has been uploaded to this site can be approximated by $f(t) = 1.1t^2 - 2.6t + 2.3$ million hours of videos per year $(0 \le t \le 9)$, where t is time in years since June 2005. Use a definite integral to estimate the total number of hours of video uploaded from June 2007 to June 2010.

Answer: 23 million hours of video

Total number of hours $= \int_2^5 f(t)\,dt = \int_2^5 1.1t^2 - 2.6t + 2.3\ dt \approx 23$ million hours of video

40. Calculate the area of the region bounded by $y = \sqrt{x}$, the x-axis, and the lines $x = 0$ and $x = 16$.

Answer: $\frac{128}{3}$

Area under curve $= \int_0^{16} \sqrt{x}\,dx = \frac{128}{3}$

Integration by Parts: $\int udv = uv - \int vdu$

41. Use integration by parts to find the integrals.
   (a) $\int (3x+4)e^{-5x}dx$
   (b) $\int x^2 \ln x\,dx$

***Answer*:**

(a) $-\frac{1}{5}(3x+4)e^{-5x} - \frac{3}{25}e^{-5x} + C = \left(-\frac{3}{5}x - \frac{23}{25}\right)e^{-5x} + C$
(Let $u = 3x+4, dv = e^{-5x}dx$. Then $du = 3dx$ and $v = -\frac{1}{5}e^{x}$)
(b) $\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$ (Let $u = \ln x\,, dv = x^2dx$)

Area between Curves

42. Find the area of the region enclosed by the curves of $f(x) = x^2 - x + 5$ and $g(x) = x + 8$.

***Answer*:** $\frac{32}{3}$

Area $= \int_{-1}^{2}(x+8) - (x^2 - x + 5)\,dx = \int_{-1}^{2} -x^2 + 2x + 3\,dx = -\frac{x^3}{3} + x^2 + 3x|_{-1}^{2} = \frac{32}{3}$

Average Value

43. Find the average value of $f(x) = 6e^{0.5x}$ over the interval $[-1, 3]$.

Answer: $3(e^{1.5} - e^{-0.5})$

The average value of a continuous function $f(x)$ over interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\frac{1}{3-(-1)}\int_{-1}^{3} 6e^{0.5x}\,dx = \frac{1}{4}\cdot 6\cdot \frac{e^{0.5x}}{0.5}\Big|_{-1}^{3} = 3e^{0.5x}\Big|_{-1}^{3} = 3(e^{1.5} - e^{-0.5})$$

44. Find the average value of the function $f(x) = 6x^2 - 4x + 7$ over the interval $[-2, 2]$.

Answer: 15

Consumers' Surplus and Producers' Surplus

45. Your video store has the exponential demand equation $p = 15e^{-0.01}$, where q represents daily sales of used DVD's and p represents daily price you charge per DVD. Calculate the daily Consumer's Surplus if you sell used DVDs at \$5 dollars each.

Answer: \$450.69

Consumer Surplus = $\int_0^{\bar{q}} D(q) - \bar{p}\,dq = \int_0^{109.9} 15e^{-0.01q} - 5\,dq = 450.69$

46. Calculate the Producer's Surplus for the supply equation $p = 13 + 2q$ at the unit price $\bar{p} = 29$.

Answer: \$64

Producer Surplus = $\int_0^{\bar{q}} \bar{p} - S(q)dq = \int_0^{8} 29 - (13 + 2q)dq = 16q - q^2\Big|_0^8 = 64$

47. Calculate the Producer Surplus for the supply equation is $p = 7 + 2q^{1/3}$ at price $\bar{p} = 14$.

Answer: \$75

$\bar{q} = 3.5^3 = 42.875$,

Producer Surplus = $\int_0^{42.875} 14 - (7 + 2q^{1/3})dq = 7q - \frac{3}{2}q^{4/3}\Big|_0^{42.875} = 75$

Improper Integral

48. Determine whether each improper integral is convergent or divergent. If it is convergent, find its value.

(a) $\int_1^\infty \frac{8}{x^2} dx$

(b) $\int_4^\infty e^{-2x} dx$

(c) $\int_1^\infty \frac{1}{x} dx$

(d) $\int_3^\infty x^2 dx$

(e) $\int_{-\infty}^0 e^{2x} dx$

Answer:

(a) Converges to 8

$$\int_1^\infty \frac{8}{x^2} dx = \lim_{t\to\infty} \int_1^t 8x^{-2}\, dx = \lim_{t\to\infty} \left(-\frac{8}{x}|_1^t\right) = \lim_{t\to\infty} \left(-\frac{8}{t} + \frac{8}{1}\right) = 0 + 8 = 8$$

(b) Converges to $\frac{1}{2e^8}$

$$\int_4^\infty e^{-2x} dx = \lim_{t\to\infty} \int_4^t e^{-2x}\, dx = \lim_{t\to\infty} \left(-\frac{1}{2} e^{-2x}|_4^t\right) = \lim_{t\to\infty} \left(-\frac{1}{2} e^{-2t} + \frac{1}{2} e^{-2(4)}\right) = 0 + \frac{1}{2} e^{-8} = \frac{1}{2e^8}$$

(c) Diverges

$$\int_1^\infty \frac{1}{x} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x} dx = \lim_{t\to\infty} (\ln t - \ln 1) = \infty - 1 = \infty$$

(d) Diverges

(e) Converges to $\frac{1}{2}$