Section 11.4 Chain Rule

1. Find the derivative \( y' \) of each function.
   (a) \( y = 0.4(3x^2 + 2x - 8)^5 \)
   (b) \( y = \sqrt{x^3 - 50x} \)
   (c) \( y = (6x + 1)^{4/3} \)
   (d) \( y = \frac{25}{(x^2+x+2)^4} \)
   (e) \( y = 8e^{-2x} \)

Answer:

(a) \( 2(3x^2 + 2x - 8)^4(6x + 2) \)
(b) \( \frac{1}{2}(x^3 - 50x)^{-\frac{1}{2}} \cdot (3x^2 - 50) = \frac{3x^2 - 50}{2\sqrt{x^3 - 50}} \)
(c) \( \frac{4}{3}(6x + 1)^{1/3} \cdot 6 = 8(6x + 1)^{1/3} \)
(d) \( 25(-4)(x^2 + x + 2)^{-5} \cdot (2x + 1) = -100(2x + 1)(x^2 + x + 2)^{-5} = -\frac{100(2x+1)}{(x^2+x+2)^5} \)
(e) \( -16e^{-2x} \)

Section 11.5 Derivative of Logarithmic and Exponential Functions

2. Find the derivative \( y' \) of each function.
   (a) \( y = 9 \ln(2x) \)
   (b) \( y = \ln(5x^3 + x^2 + 4) \)
   (c) \( y = \ln|x^3 - 8x| \)
   (d) \( y = x - x \ln x \)
   (e) \( y = 4e^{x^2-3x} \)
   (f) \( y = (x^2 - 2x)e^{2x+3} \)
   (g) \( y = e^{5/x} \)

Answer:

(a) \( 9 \cdot \frac{1}{2x} \cdot 2 = \frac{9}{x} \)
(b) \( \frac{15x^2+2x}{5x^3+x^2+4} \)
(c) \( \frac{3x^2-8}{x^3-8x} \)
(d) \( 1 - \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) = -\ln x \)
(e) \( 4xe^{x^2-3x} \cdot (5x^4 - 3) \)
(f) \( (2x - 2)e^{2x+3} + (x^2 - 2x)e^{2x+3} \cdot 2 = (2x^2 - 2x - 2)e^{2x+3} \)
(g) \(e^{5/x} \cdot (-5x^{-2})\)

**Section 11.6 Implicit Differentiation**

3. Find the derivative \(\frac{dy}{dx}\).
   (a) \(x^3 - y^3 + y = 3\)
   (b) \(6x^2 y - 15x = y^2\)
   (c) \(xe^y - e^x = 0\)

*Answer:*

(a) \(\frac{dy}{dx} = \frac{2x^2}{3y^2 - 1}\)
(b) \(\frac{dy}{dx} = \frac{15 - 12xy}{6x^2 - 2y}\)
(c) \(\frac{dy}{dx} = \frac{e^x - e^y}{xe^y}\)

**Section 12.1 Maxima and Minima**

4. The function of a function \(f\) on \([-3, 3]\) is given below.
   (a) \(f\) has a relative minimum at \(x =\)
   (b) \(f\) has an absolute maximum at \(x =\)
   (c) \(f\) has an absolute minimum at \(x =\)
   (d) \(f'\) is zero at \(x =\)
   (e) \(f'\) is positive on interval(s):
   (f) \(f'\) is negative on interval(s):

*Answer:*

(a) 0
(b) \(-1, 1\)
(c) \(-3, 3\)
(d) \(-1, 0, 1\)
(e) \((-3, -1)\) and \((0, 1)\)
(f) \((-1, 0)\) and \((1, 3)\)
5. The graph of a function $f$ is given below.
   (a) $f'$ is zero at $x =$
   (b) $f$ has a relative max at $x =$ and has a relative min at $x =$.

\[ \begin{align*}
\text{Answer:} \\
(a) & \ -2, 0, 2 \\
(b) & \ -2; 2 
\end{align*} \]

6. Find all critical points of the function. Use the First Derivative Test to determine whether $f$ has a relative minimum, a relative maximum or neither at the critical point.
   $f(x) = x - 2 \ln x$, $x > 0$

   \text{Answer:} Critical point $x = 2$; $f$ has a relative minimum at $x = 2$, which is equal to $f(2) = 2 - 2 \ln 2$.

7. Consider the function $f(x) = 8x^3 - 24x + 12$.
   (a) Find all critical points of $f$.
   (b) Find the absolute extrema of $f$ over interval $[-3, 2]$.

   \text{Answer:} (a) Critical points: $x = -1, 1$.
   \hspace{1cm} (b) Abs Max = $f(-1) = f(2) = 28$; Abs Min= $f(-3) = -132$.

8. Consider the function $g(x) = (x - 2)^{2/3}$.
   (a) Find any critical points of $g$.
   (b) Find the absolute max and absolution min of $g$ over $[0, 5]$.

   \text{Answer:} (a) Critical point: $x = 2$
   \hspace{1cm} (b) Abs Max = $g(5) = 3^{2/3}$; Abs Min= $g(2) = 0$. 

\[ \text{Page 3} \]
9. For the function \( k(x) = 4x^3 - 24x^2 + 36x - 20 \),
(a) Find any critical points of \( k \).
(b) For what \( x \) values is the function \( k \) increasing? decreasing?
(c) Find any relative and absolute extrema of \( k \) on \([-1, 5]\).

**Answer:**
(a) Critical points: \( x = 1, 3 \).
(b) Increasing on \((-1, 1) \cup (3, 5)\); decreasing on \([1, 3]\).
(c) Abs Max \( k(5) = 60 \); Abs Min \( k(-1) = -84 \); Rel Max \( k(1) = -4 \); Rel Min \( k(3) = -20 \).

10. For the function \( h(x) = e^x - x \),
(a) Find any critical points of \( h \).
(b) Find any relative and absolute extrema of \( h \) on \([-2, 3]\).

**Answer:**
(a) Critical points: \( x = 0 \).
(b) Abs Max \( h(3) = e^3 - 3 \); Abs Min \( h(0) = 1 \); Rel Max \( h(-2) = e^{-2} + 2 \); Rel Min: None.

11. Suppose \( f(x) \) is continuous on \((-\infty, \infty)\) and \( f \) has two critical points at \( x = -1 \) and \( x = 2 \). If we know \( f'(-2) < 0, f'(0) > 0, \) and \( f'(3) < 0 \), determine whether each statement is True or False.
(a) T or F \( f \) has a relative minimum at \( x = -1 \) because \( f \) is decreasing on the left side of \( x = -1 \) and increasing on the right side of \( x = -1 \).
(b) T or F \( f \) has a relative maximum at \( x = 2 \) because \( f' \) is positive on the left side of \( x = 2 \) and negative on the right side of \( x = 2 \).
(c) T or F \( f \) is decreasing on the interval \([-1, 2]\).
(d) T or F \( f \) is decreasing on the interval \((2, \infty)\).

**Answer:** T, T, F, T.

12. Suppose \( f(x) \) is continuous on \((-\infty, \infty)\) and \( f \) has two critical points at \( x = 0 \) and \( x = 4 \). If \( f'(-1) < 0, f'(1) < 0, \) and \( f'(5) > 0 \), then
(a) \( f \) has ________ (relative minimum/relative minimum/no relative extrema) at \( x = 0 \).
(b) \( f \) has ________ (relative minimum/relative minimum/no relative extrema) at \( x = 4 \).
(c) \( f \) is increasing on interval(s):
(d) \( f \) is decreasing on interval(s):

**Answer:**
(a) No relative extrema
(b) Relative min
(c) \((4, \infty)\)
(d) \((-\infty, 4)\)
13. You are running a business selling homemade bread. Your weekly revenue from the sale of \( q \) loaves bread is \( R(q) = 68q - 0.1q^2 \) dollars, and the weekly cost of making \( q \) loaves of bread is \( C(q) = 23 + 20q \).
   (a) Find the weekly profit function \( P(q) \).
   (b) Find the production level \( q \) that maximizes the weekly profit.
   (c) Find the maximum profit.

   **Answer:**
   (a) \( P(q) = 68q - 0.1q^2 - (23 + 20q) = -0.1q^2 + 48q - 23 \)
   (b) Set \( P' = -0.2q + 48 = 0 \), and solve for \( q \): \( q = 240 \)
   (c) \( P(240) = -0.1 \cdot 240^2 + 48 \cdot 240 - 23 = 5737 \) dollars.

14. Suppose \( C(x) = 0.02x^2 + 2x + 4000 \) is the total cost for a company to produce \( x \) units of a certain product. Find the production level \( x \) that minimizes the average cost \( \bar{C}(x) = \frac{C(x)}{x} \).

   **Answer:** \( \bar{C}(x) = 0.02x + 2 + 4000x^{-1} \). Set \( \bar{C}'(x) = 0.02 - 4000x^{-2} = 0 \) and solve for \( x \): \( x = 447 \).

15. I would like to create a rectangular orchid garden that abuts my house so that the house itself forms the northern boundary. The fencing for the southern boundary costs $3 per foot, and the fencing for the east and west sides costs $5 per foot. If I have a budget of $120 for the project, what are the dimensions of the garden with the largest area I can enclose?

   **Answer:** Maximize area \( A = xy \) subject to cost \( 3x + 10y = 120 \), where \( x \) is the length of the north and south sides, and \( y \) is the length of east and west sides.

   Dimension is \( x = 20, y = 6 \). Max area = 120 square feet.

16. I need to create a rectangular vegetable patch with an area of exactly 162 square feet. The fencing for the east and west sides costs $4 per foot, and the fencing for the north and south sides costs only $2 per foot. What are the dimensions of the vegetable patch with the least expensive fence?

   **Answer:** Minimize the cost \( C = 4x + 8y \) subject to area \( xy = 162 \), where \( x \) is the length of the north and south sides, and \( y \) is the length of east and west sides.

   Dimension is \( x = 18, y = 9 \). Minimum cost = 144 dollars.
17. Worldwide annual sale of a product in 2013-2017 were projected to be approximately \( q = -10p + 4220 \) million units at a selling price of \( p \) dollars per unit. What selling price would have resulted in the largest projected annual revenue? What would be resulting revenue?

**Answer:** \( R(p) = p \cdot q = p(-10p + 4220) = -10p^2 + 4220p \). Set \( R'(p) = -20p + 4220 = 0 \) and solve for \( p \): \( p = 211 \). Maximum revenue is \( R(211) = 211 \cdot 2110 = 445210 \)

Section 12.3 Higher-order Derivatives, Acceleration and Concavity

18. Find the second derivative \( y'' \) for each function.

(a) \( y = 2e^{2x-5} \)
(b) \( y = \frac{7}{x} - 5 \ln x \)

**Answer:**

(a) \( y'' = 8e^{2x-5} \)
(b) \( y'' = 14x^{-3} + 5x^{-2} \)

19. The graph of a function \( y = f(x) \) is given below.

(a) \( f'(-4) = f'(0) = \)
(b) Is \( f''(-4) \) is positive or negative? Is \( f''(0) \) is positive or negative?
(c) If \( f \) has a point of inflection at \( x = -2 \), then \( f''(-2) = \)
(d) \( f \) is concave_______ (up/down) on interval \((-\infty, -2)\), and concave_______ (up/down) on interval \((-2, \infty)\).

**Answer:** (a) 0; (b) negative, positive; (c) 0; (d) down, up.
20. The graph of a function $f(x)$ is given. Fill in the blank.
(a) The graph is concave _______ (up/down) on interval $(-\infty, 2)$, concave _______ (up/down) on interval $(2, 3)$, and concave _______ (up/down) on interval $(3, \infty)$.
(b) The second derivative $f''$ is positive on: _______________ and negative on: _______________.
(c) List the points of inflection: $x =$
(d) Does $f$ have any relative extrema?
(e) Does $f$ have any absolute extrema?

![Graph of a function](image)

**Answer:**
(a) down, up, down
(b) $(2, 3)$, $(-\infty, 2) \cup (3, \infty)$;
(c) $x = 2, 3$
(d) None
(e) Abs max at $x = 3.5$, no abs min.

21. Suppose the position of a particle moving on a straight line is $s(t) = \sqrt{t} + 4t^2$. Find the particle’s acceleration as a function of time $t$.

**Answer:** $a(t) = s''(t) = -\frac{1}{4}t^{-\frac{3}{2}} + 8$

22. Let $s(t) = 4e^t - 8t^2 + 3$ be the position function of a particle moving in a straight line, where $s$ is measured in feet and $t$ is measured in seconds. Find its acceleration when $t = \ln 6$ seconds.

**Answer:** $8$ ft/sec$^2$
Section 12.5 Related Rates

23. The radius of a circular puddle is growing at a rate of 15 cm/sec.
   (a) How fast is its area growing at the instant when the radius is 30 cm?
   (b) How fast is the area growing when the area is 81 square centimeters?

   Answer: (a) 2827 cm²/sec; (b) 479 cm²/sec

24. A rather flimsy spherical balloon is designed to pop at the instant its radius has reached 6 cm. Assuming the balloon is filled with helium at a rate of 13 cubic centimeters per second, calculate how fast the radius is growing at the instant it pops.

   Answer: 0.03 cm/sec

25. The average cost for the weekly manufacture of retro portable CD player is given by

   \[ C(x) = 120,000x^{-1} + 20 + 0.0004x \]

   where \( x \) is the number of CD players manufactured that week. Weekly production is currently 4,000 players and is increasing at a rate of 400 players per week. What is happening to the average cost? Fill in the blank.

   The average cost is \( \) increasing/decreasing at a rate of \( \) dollars per player per week.

   Answer: When \( x = 4000 \), \( \frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt} = -2.84 \). So, the average cost is decreasing at a rate of 2.84 dollars per player per week.

Section 12.6 Elasticity

26. Suppose the elasticity of demand is 3.2, when the price of a product is $25. This means the demand is going up/down by \( \) % for 1% increase in the price. A small increase in price will result in a increase/decrease in the revenue.

   Answer: down, 3.2%, decrease.

27. Suppose the elasticity of demand is 0.65, when the price of a product is $500. This means the demand will go up/down by \( \) % for 1% increase in the price. A small increase in price will cause the revenue to increase/decrease.

   Answer: down, 0.65%, increase.
28. The weekly sales of some backpacks is given by \( q = 1080 - 18p \), where the \( q \) represents the quantity of backpacks sold at price \( p \).
   (a) Find the elasticity of demand at the price of $20. Interpret your answer.
   (b) Is the demand at the price $20 elastic, inelastic, or unit elastic? Should the price be raised or lowered from $20 to increase the revenue?
   (c) What price will maximize the revenue?
   (d) What is the maximum weekly revenue?

   Answer:
   (a) \( E(p) = -(-18) \cdot \frac{p}{1080-18p} = \frac{18p}{1080-18p} \), so \( E(20) = \frac{18 \cdot 20}{1080-18 \cdot 20} = 0.5 \).
   This means the demand will drop by 0.5% for 1% increase from current price $20.
   (b) 0.5 < 1, it is inelastic. The price should be raised to increase revenue.
   (c) Solve for the price when \( E(p) = 1 \). Solving \( \frac{18p}{1080-18p} = 1 \) gives \( p = $30 \).
   (d) \( R = pq = 30(1080 - 18 \cdot 30) = 16200 \) dollars.

29. Suppose the demand function is \( q = -2p^2 + 33p \), where \( q \) represents the quantity sold at price \( p \).
   (a) Find the price elasticity of demand \( E(p) \).
   (b) Find the elasticity when \( p = $15 \). If the price increase by 1%, the demand will drop by how much? Should the price be lowered or raised from $15 to increase the revenue?

   Answer:
   (b) \( E(p) = -(-4p + 33) \cdot \frac{p}{-2p^2 + 33p} = \frac{4p - 33}{-2p + 33} \)
   (c) \( E(15) = \frac{4(15) - 33}{-2(15) + 33} = \frac{27}{3} = 9 > 1 \). It is elastic. The demand will drop by 9% if the price increases by 1%.
   The price should be lowered from $15 to increase revenue.