INSTRUCTIONS

Write your solutions on the blank pages provided. Write on only one side of each page, and leave reasonable margins. (Papers will be photocopied before grading.) Put your name on each page that you submit, and number the pages sequentially.

DO NOT STAPLE OR FOLD THE PAGES

Turn in this cover sheet and the exam problem page with your solutions. If you are enrolled in MAT 571, the exam will automatically be graded as the final exam. Be sure to check the box (below) on this cover sheet if you want it to be graded as a qualifying exam.

No notes, books, calculators, or outside assistance is allowed.

Be sure to read and follow the instructions for each problem. You may use any part of a problem (solved or not) in the solution of any other problem, and in any later part of the same problem. Unless otherwise indicated, you may use any results from the course lectures, from the text (through Section 3.3 of Folland), and from any homework you turned in. When you use such results, be sure to make some acknowledgement: for example, “since $\mathbb{Q}$ is dense in $\mathbb{R}$, . . .” or “by the Algebra of Limits . . .”

NAME:________________________________________________________________________

EMAIL:________________________________________________________________________

☐ CHECK THIS BOX IF YOU WANT TO COUNT THIS AS A QUALIFYING EXAM

Date: April 26, 2012.
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Problem 1. In each part, give an example with the stated property, or briefly explain why no such example exists.
(a) A finite Borel measure on \( \mathbb{R} \).
(b) A Lebesgue-measurable function \( f \) on \( \mathbb{R}^2 \) such that \( f_x \) is not measurable for some \( x \in \mathbb{R} \), where by definition \( f_x(y) = f(x, y) \) for \( y \in \mathbb{R} \).
(c) Nonzero complex measures \( \xi, \eta, \) and \( \mu \) such that \( \xi \perp \eta, \eta \perp \mu, \) and \( \mu \perp \xi \).

Problem 2. Let \( \mu \) be a Lebesgue-Stieltjes measure on \( \mathbb{R} \) with domain \( \mathcal{M} \), and suppose \( E \in \mathcal{M} \) with \( \mu(E) < \infty \). Using the fact that
\[
\mu(E) = \inf \{ \mu(U) \mid E \subseteq U \text{ and } U \text{ is open} \},
\]
prove that for each \( \epsilon > 0 \) there exists a set \( A \) which is a finite disjoint union of open intervals such that \( \mu(E \triangle A) < \epsilon \). (Recall that \( E \triangle A = (E \setminus A) \cup (A \setminus E) \).)

Problem 3. Find the limit, and justify your answer:
\[
\lim_{n \to \infty} \int_0^{\infty} e^{-x} \sin^2(x/n) \, dx.
\]

Problem 4. Let \( \mu \) be a \( \sigma \)-finite (positive) measure on a measurable space \( (X, \mathcal{M}) \), and for each \( n \in \mathbb{N} \) let \( \nu_n \) be a positive measure on \( (X, \mathcal{M}) \) such that \( \nu_n \ll \mu \).
(a) Prove that \( \nu \ll \mu \), where \( \nu = \sum \nu_n \).
(b) Assuming that \( \nu = \sum \nu_n \) is \( \sigma \)-finite, prove that
\[
\frac{d\nu}{d\mu} = \sum_{n=1}^{\infty} \frac{d\nu_n}{d\mu} \quad \mu\text{-a.e.}
\]

Problem 5. For any \( E \subseteq \mathbb{R} \), define \( d(E) \subseteq \mathbb{R}^2 \) by
\[
d(E) = \{(x, y) \mid x - y \in E\}.
\]
Prove that \( d(E) \) is Lebesgue measurable whenever \( E \) is Lebesgue measurable.