Answer as many questions as you can, attempting at least one question from problems 1 to 4, and at least one question from problems 5 to 8. Complete solutions to fewer problems are preferable to fragmentary solutions to all problems.

1. Suppose that $G$ is a group.
   (a) (15 pts) Let $a, b \in G$. Prove that the order of $ab$ equals the order of $ba$.
   (b) (10 pts) Suppose $n > 1$ is an integer such that $G$ has a unique element $g$ of order $n$. Show $n = 2$, and that $g \in Z(G)$, the center of $G$.

2. Let $G$ be a finite Abelian group of order $p_1^{r_1}p_2^{r_2} \cdots p_t^{r_t}$ for distinct primes $p_i$. Let $G(p_i)$ denote the elements of $G$ that have order a power of $p_i$.
   (a) (10 pts) Prove that $G(p_i)$ is a subgroup of $G$.
   (b) (10 pts) Prove that $G = G(p_1) \oplus G(p_2) \oplus \cdots \oplus G(p_t)$.
   (c) (5 pts) Prove that $G$ is the direct sum of its Sylow $p_i$-subgroups for $i = 1, \ldots, t$.

3. (a) (8 pts) Let $G_1$ be the group (under ordinary matrix multiplication) generated by the complex matrices
   \[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \]
   where $i^2 = -1$. Prove that $G_1$ is a non-Abelian group of order 8.
   (b) (8 pts) Let $G_2$ be the group (under ordinary matrix multiplication) generated by the real matrices
   \[ C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]
   Prove that $G_2$ is a non-Abelian group of order 8.
   (c) (9 pts) Are the groups $G_1, G_2$ of parts (a) and (b) isomorphic? Justify your answer.

4. (a) (5 pts) Determine (up to isomorphism) the Abelian groups of order 147.
   (b) (5 pts) Prove that every group of order 147 has a normal Sylow 7-subgroup.
   (c) (15 pts) Prove that there is a unique non-Abelian group of order 147 whose 7-Sylow subgroup is cyclic.
   [You may assume Sylow’s Theorems].
5. Let $R$ be a Noetherian ring with unity.

(a) (4 pts) Prove that maximal ideals are prime.

(b) (4 pts) Prove that the union of all maximal ideals of $R$ is the set of non-units of $R$.

(c) (4 pts) Prove that $R$ has a unique maximal ideal if and only if the set of non-units of $R$ is an ideal.

(d) (5 pts) Prove that the union of all prime ideals of $R$ is the set of non-units of $R$.

(e) (8 pts) Prove that the intersection of all prime ideals is the set of nilpotent elements of $R$.

6. Let $R$ be a commutative ring containing a field $F$ as subring. Suppose further that $R$ is a 2-dimensional vector space over $F$.

(a) (5 pts) Prove there exists $a \in R$ such that $R = F[a]$.

(b) (6 pts) Prove that $R \simeq F[x]/(p(x))$ where $p(x) \in F[x]$ is of degree 2.

(c) (6 pts) Prove that either $R$ is a field or $R \simeq F \times F$ or $R \simeq F[x]/(x^2)$.

(d) (8 pts) Prove that $F \times F$ and $F[x]/(x^2)$ are not isomorphic, and neither is a field.

7. (a) (10 pts) Prove that $K_1 = \mathbb{F}_7[x]/(x^2 + 1)$ and $K_2 = \mathbb{F}_7[y]/(y^2 + 6y + 3)$ are both fields with 49 elements.

(b) (15 pts) Determine an explicit isomorphism (defined by $\bar{x} \to f(\bar{y})$) from $K_1$ to $K_2$.

8. (a) (6 pts) Prove that in an integral domain $D$ a prime element is irreducible.

(b) (6 pts) Give an example (with justification) of an irreducible element in an integral domain that is not prime.

(c) (6 pts) Prove that in a Principal Ideal Domain a nonzero element is prime if and only if it is irreducible.

(d) (7 pts) Give an example of an integral domain that is not a Unique Factorization Domain.