MAT 512 Final Exam
Combinatorics Qualifier Exam
December 17, 2012

Notation. Let \([n] = \{1, 2, \ldots, n\}\). For a finite set \(X\) denote by \(\binom{X}{k}\), \(\{X\}_k\), \([X]_k\), and \(X!\) the sets of all subsets of \(X\) of size \(k\), partitions of \(X\) into \(k\) parts, permutations of \(X\) into \(k\) cycles, and permutations of \(X\), respectively. When \(|X| = n\) these sets have cardinalities \(\binom{n}{k}\), \(\{n\}_k\), \([n]_k\), and \(n!\), respectively. Define \(x^n = x(x+1) \cdots (x+n-1)\) and \(x^n = x(x-1) \cdots (x-n+1)\).

Directions. Solve the 6 problems below, each problem on a separate sheet. Show your work and explain your reasoning. Ask questions if you find the wording ambiguous or confusing. You may leave your answers in the forms of the notations above without calculating their values or displaying their formulas.

1. Let \(B\) be an \(m \times n\) chess-like board and \(S\) be the set of all black and white colorings of the squares of \(B\). A coloring in \(S\) is \(C\)-good if no column has a black square below a white square, and it is \(R\)-good if no row has a black square to the right of a white square. Determine the number of:
   (a) \(C\)-good colorings.
   (b) \(C\)-good colorings such that no two columns have the same number of black squares.
   (c) \(C\)-good colorings such that no two columns have the same number of black squares and, for each \(i\), the \(i\)th column does not have exactly \(i\) black squares.
   (d) colorings that are both \(C\)-good and \(R\)-good.
   (e) colorings that are both \(C\)-good and \(R\)-good and adjacent columns are not identical.

2. Prove that \(\binom{r+s+1}{r} = \sum_{t=0}^{r} (s+t) \binom{s+t}{t}\) for all \(r, s \geq 0\).

3. Use combinatorial reasoning to prove that \(k! \{n\}_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n\).

4. Let \(a_0, a_1, a_2, \ldots\) be the sequence defined by \(a_0 = 1\), \(a_1 = 2\), and \(a_n = 2a_{n-1} - a_{n-2} + 3\) for \(n \geq 2\). Derive a closed formula for \(a_n\).

5. Let \(B_n\) denote the number of partitions of \([n]\).
   (a) Prove that \(B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i\) for all \(n \geq 0\).
   (b) Use this recursion to find a simple expression for the exponential generating function for the Bell numbers \(\{B_n\}_{n \geq 0}\).

6. Let \(\Pi_n\) denote a solid 3-dimensional polytope with regular \(n\)-gon \(B\) as base and vertex directly above the center of the \(n\)-gon, at which \(n\) identical triangles meet, each based at the edges of \(B\). For example, a regular tetrahedron is in the shape of a \(\Pi_3\) and an Egyptian pyramid is in the shape of a \(\Pi_4\).
   (a) Write the cycle index polynomial \(P(z)\) for the symmetry group of the faces of a \(\Pi_{15}\).
   (b) Use \(P(z)\) to count the number of inequivalent 2-colorings of the faces of a \(\Pi_{15}\).
   (c) Use \(P(z)\) to count the number of inequivalent 3-colorings of the faces of a \(\Pi_{15}\) in which one face is red, five faces are blue, and ten faces are green.