PRACTICE PROBLEMS CHAPTER 6 AND 7

I. Laplace Transform

1. Find the Laplace transform of the following functions.

(a) $f(t)=\sin(2t)\cos(2t)$

(b) $f(t)=\cos^2(3t)$

(c) $f(t)=t\,e^{2t}\sin(3t)$

(d) $f(t)=(t+3)u_7(t)$

(e) $f(t)=t^2u_3(t)$

(f) $f(t)=\begin{cases} 1, & \text{if } 0\le t<2, \\ t^2-4t+4, & \text{if } t\ge 2 \end{cases}$

(g) $f(t)=\begin{cases} t, & \text{if } 0\le t<3, \\ 5, & \text{if } t\ge 3 \end{cases}$

(h) $f(t)=\begin{cases} 0, & \text{if } t<\pi, \\ t-\pi, & \text{if } \pi\le t<2\pi \\ 0, & \text{if } t\ge 2\pi \end{cases}$

(i) $f(t)=\begin{cases} \cos(\pi t), & \text{if } t<4, \\ 0, & \text{if } t\ge 4 \end{cases}$

(j) $f(t)=\begin{cases} t, & \text{if } 0\le t<1, \\ e^t, & \text{if } t\ge 1 \end{cases}$

2. Find the inverse Laplace Transform:

(a) $F(s)=\dfrac{1}{(s+1)(s^2-1)}$

(b) $F(s)=\dfrac{2s+3}{s^2+4s+13}$

(c) $F(s)=\dfrac{e^{-3s}}{s-2}$

(d) $F(s)=\dfrac{1+e^{-2s}}{s^2+6}$

3. The transform of the solution to a certain differential equation is given by $X(s)=\frac{1-e^{-2\pi s}}{s^2+1}$.

Determine the solution $x(t)$ of the differential equation.

4. Suppose that the function $y(t)$ satisfies the DE $y''-2y'-y=1$, with initial values, $y(0)=-1,\ y'(0)=1$. Find the Laplace transform of $y(t)$

5. Consider the following IVP: $y''-3y'-10y=1,\quad y(0)=-1, y'(0)=2$

(a) Find the Laplace transform of the solution $y(t)$.

(b) Find the solution $y(t)$ by inverting the transform.

6. Consider the following IVP: $y''+4y=4u_5(t),\quad y(0)=0, y'(0)=1$

(a) Find the Laplace transform of the solution $y(t)$.

(b) Find the solution $y(t)$ by inverting the transform.

7. A mass $m=1$ is attached to a spring with constant $k=5$ and damping constant $c=2$. At the instant $t=\pi$ the mass is struck with a hammer, providing an impulse $p=10$. Also, $x(0)=0$ and $x'(0)=0$.

a) Write the differential equation governing the motion of the mass.

b) Find the Laplace transform of the solution $x(t)$.

c) Apply the inverse Laplace transform to find the solution.

II. Linear systems

1. Verify that $\mathbf{x}=e^t\begin{pmatrix}1\\0\end{pmatrix}+2te^t\begin{pmatrix}1\\1\end{pmatrix}$ is a solution of the system $\mathbf{x}'=\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\mathbf{x}+e^t\begin{pmatrix}1\\-1\end{pmatrix}$

2. Given the system $x'=tx-y+e^tz,\quad y'=2x+t^2y-z,\quad z'=e^{-t}+3ty+t^3z$, define $\mathbf{x}$, P(t) and $\mathbf{f}(t)$ such that the system is represented as $\mathbf{x}'=\mathrm{P}(t)\mathbf{x}+\mathbf{f}(t)$

3. Consider the second order initial value problem: $u''+2u'+2u=3\sin t,\quad u(0)=2, u'(0)=-1$

Change the IVP into a first-order initial value system and write the resulting system in matrix form.

4. Are the vectors $\mathbf{x}_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\mathbf{x}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\mathbf{x}_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$ linearly independent?

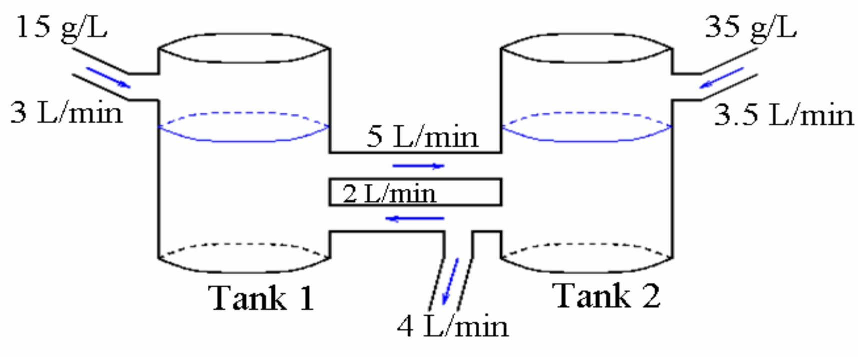
5. Consider the system $\mathbf{x}'=\begin{pmatrix}-2 & -6\\0 & 1\end{pmatrix}\mathbf{x}$

Two solutions of the system are $\mathbf{x}_1=e^t\begin{pmatrix}-2\\1\end{pmatrix}$ and $\mathbf{x}_2=e^{-2t}\begin{pmatrix}1\\0\end{pmatrix}$

(a) Use the Wronskian to verify that the two solutions are linearly independent.

(b) Write the general solution of the system.

6. Consider two interconnecting tanks as shown in the figure. Tank 1 initially contains 80 L (liters) of water and 100 g (grams) of salt, while Tank 2 initially contains 20 L of water and 50 g of salt. Water containing 15g/L of salt is poured into tank 1 at a rate of 3 L/m while the mixture flowing into tank 2 contains a salt concentration of 35 g/L and is flowing at a rate of 3.5 L/min. The mixture flows from tank 1 to tank 2 at a rate of 5 L/min. The mixture drains from tank 2 at a rate of 6 L/min, of which some flows back into Tank 1 at a rate of 2 L/min, while the remainder leaves the tank.
Let Q_1 and Q_2, respectively, be the amount of salt in each tank at time t.
Write down differential equations and initial conditions that model the flow process.



7. Suppose the system $\mathbf{x}'=A\mathbf{x}$ has the general solution $\mathbf{x}(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}=c_1 e^{t}\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}+c_2 e^{-2t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+c_3 e^{-t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Given the initial condition $\mathbf{x}(0)=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, find $x_1(t), x_2(t)$ and $x_3(t)$.

8. Solve the IVP $\mathbf{x}'=A\mathbf{x}$ with $A=\begin{pmatrix} 1 & -3 \\ 0 & -2 \end{pmatrix}$ and $\mathbf{x}(0)=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

9. Solve the IVP $\begin{aligned} x' &= x+2y \\ y' &= 4x+3y \end{aligned}$ with $x(0)=3,\ y(0)=0$.

10. Suppose that A is a real 3×3 matrix that has the following eigenvalues and eigenvectors

$$-2,\ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\quad 1+i,\ \begin{pmatrix} 1-i \\ 2 \\ 1 \end{pmatrix},\quad 1-i,\ \begin{pmatrix} 1+i \\ 2 \\ 1 \end{pmatrix}$$

Find a fundamental set of real valued solutions to the system $\mathbf{x}'=A\mathbf{x}$.

11. Solve the initial value problem $x_1'=x_1-2x_2,\ \ x_2'=2x_1+x_2,\ \ x_1(0)=0, x_2(0)=4$ using the eigenvalue method. Express the solution in terms of real functions only (no complex functions).

ANSWERS TO PRACTICE PROBLEMS CHAPTER 6 AND 7

I. Laplace Transform

1. (a) Using the double angle trigonometric identity, the function $f(t)$ can be rewritten as $f(t)=\frac{1}{2}\sin(4t)$. Thus $\mathcal{L}\{f(t)\}=\frac{2}{s^2+16}$

(b) Using the half angle trigonometric identity, the function $f(t)$ can be rewritten as $f(t)=\frac{1}{2}(1+\cos(6t))$. Thus $\mathcal{L}\{f(t)\}=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+36}\right)$

(c) Using the property $\mathcal{L}\{t f(t)\}=-F'(s)$ with $F(s)=\mathcal{L}\{e^{2t}\sin(3t)\}=\frac{3}{(s-2)^2+9}$ yields $\mathcal{L}\{t e^{2t}\sin(3t)\}=\frac{6(s-2)}{\left((s-2)^2+9\right)^2}$

(d) $f(t)=[(t-7)+10]u_7(t)$. Thus $\mathcal{L}\{f(t)\}=e^{-7s}\mathcal{L}\{t+10\}=e^{-7s}\left(\frac{1}{s^2}+\frac{10}{s}\right)$

(e) $\mathcal{L}\{f(t)\}=e^{-3s}\mathcal{L}\{(t+3)^2\}=e^{-3s}\mathcal{L}\{t^2+6t+9\}=e^{-3s}\left(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}\right)$

(f) $f(t)=1+u_2(t)(t^2-4t+3)=1+u_2(t)\left[(t-2)^2-1\right]$

Thus $\mathcal{L}\{f(t)\}=\frac{1}{s}+e^{-2s}\mathcal{L}\{t^2-1\}=\frac{1}{s}+e^{-2s}\left(\frac{2}{s^3}-\frac{1}{s}\right)$

(g) $f(t)=t-u_3(t)(t-5)=t-u_3(t)[(t-3)-2]$. Thus

$\mathcal{L}\{f(t)\}=\frac{1}{s^2}-e^{-3s}\mathcal{L}\{t-2\}=\frac{1}{s^2}-e^{-3s}\left(\frac{1}{s^2}-\frac{2}{s}\right)$

(h) $f(t)=u_\pi(t)(t-\pi)-u_{2\pi}(t)(t-\pi)=u_\pi(t)(t-\pi)-u_{2\pi}(t)((t-2\pi)+\pi)$

Thus $\mathcal{L}\{f(t)\}=e^{-\pi s}\mathcal{L}\{t\}-e^{-2\pi s}\mathcal{L}\{t+\pi\}=\frac{e^{-\pi s}}{s^2}-e^{-2\pi s}\left(\frac{1}{s^2}+\frac{\pi}{s}\right)$

(i) $f(t)=\cos(\pi(t))-u_4(t)\cos(\pi t)=\cos(\pi(t))-u_4(t)\cos(\pi(t-4))$ Thus

$\mathcal{L}\{f(t)\}=\frac{s}{\pi^2+s^2}-e^{-4s}\mathcal{L}\{\cos(\pi t)\}=\frac{s}{\pi^2+s^2}-e^{-4s}\frac{s}{\pi^2+s^2}$

(j) $f(t)=t+u_1(t)[e^t-t]=t+u_1(t)[e^{(t-1)+1}-(t-1)-1]$ Thus

$\mathcal{L}\{f(t)\}=\frac{1}{s^2}+e^{-s}\mathcal{L}\{e^{t+1}-t-1\}=\frac{1}{s^2}+e^{-s}\left(\frac{e}{s-1}-\frac{1}{s^2}-\frac{1}{s}\right)$

2.

(a) Using PFD, $F(s)=-\frac{1}{4}\frac{1}{s+1}-\frac{1}{2}\frac{1}{(s+1)^2}+\frac{1}{4}\frac{1}{s-1}$. Thus $f(t)=-\frac{1}{4}e^{-t}-\frac{1}{2}te^{-t}+\frac{1}{4}e^t$

(b) $F(s)$ can be rewritten as $F(s)=\frac{2s+3}{(s+2)^2+9}=\frac{2(s+2)-1}{(s+2)^2+9}=\frac{2(s+2)}{(s+2)^2+9}-\frac{1}{3}\frac{3}{(s+2)^2+9}$.

Thus $f(t)=e^{-2t}\left(2\cos 3t-\frac{1}{3}\sin 3t\right)$

(c) The inverse Laplace is $u_3(t)f(t-3)$ where $f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$.

Thus $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s-2}\right\}=u_3(t)e^{2(t-3)}$

(d) $F(s)=\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{s^2+6}+\frac{e^{-2s}}{\sqrt{6}}\frac{\sqrt{6}}{s^2+6}$ thus $\mathcal{L}^{-1}\{F(s)\}=\frac{1}{\sqrt{6}}\sin(\sqrt{6}t)+\frac{1}{\sqrt{6}}u_2(t)\sin(\sqrt{6}(t-2))$

3. $(1-u_{2\pi}(t))\sin t$

4. $Y(s)=\frac{-s+3}{s^2-2s-1}+\frac{1}{s(s^2-2s-1)}$

5. (a) $Y(s)=\frac{1}{s(s-5)(s+2)}-\frac{1}{s+2}$. (b) $y(t)=-\frac{1}{10}+\frac{1}{35}e^{5t}-\frac{13}{14}e^{-2t}$

6. (a) $Y(s)=\frac{1}{s^2+4}+e^{-5s}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$. (b) $y(t)=\frac{1}{2}\sin(2t)+u_5(t)[1-\cos(2t-10)]$

7. (a) $x''+2x'+5x=10\delta(t-\pi)$ (b) $X(s)=\frac{10e^{-\pi s}}{s^2+2s+5}=5e^{-\pi s}\frac{2}{(s+1)^2+4}$

(c) $x(t)=5u_\pi(t)e^{-(t-\pi)}\sin(2(t-\pi))=5u_\pi(t)e^{-t+\pi}\sin(2t)$

II. Linear Systems

1. Differentiating the given **x** yields $\mathbf{x}'=e^t\begin{pmatrix}1\\0\end{pmatrix}+(2e^t+2te^t)\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3e^t+2te^t\\2e^t+2te^t\end{pmatrix}$

   Substituting **x** into the right hand side of the DE yields:

$$\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}e^t+2te^t\\2te^t\end{pmatrix}+e^t\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}2e^t+4te^t-2te^t\\3e^t+6te^t-4te^t\end{pmatrix}+\begin{pmatrix}e^t\\-e^t\end{pmatrix}=\begin{pmatrix}3e^t+2te^t\\2e^t+2te^t\end{pmatrix}=\mathbf{x}'$$

2. $\mathbf{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ $\quad \mathrm{P}(t)=\begin{pmatrix}t & -1 & e^t\\2 & t^2 & -1\\0 & 3t & t^3\end{pmatrix}$ $\quad \mathbf{f}(t)=\begin{pmatrix}0\\0\\e^{-t}\end{pmatrix}$

3. $\begin{pmatrix}u'\\v'\end{pmatrix}=\begin{pmatrix}0 & 1\\-2 & -2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}0\\3\sin t\end{pmatrix}$ $\quad \begin{pmatrix}u(0)\\v(0)\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}$

4. $c_1\begin{pmatrix}1\\-1\\1\end{pmatrix}+c_2\begin{pmatrix}0\\1\\1\end{pmatrix}+c_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ yields $\begin{matrix}c_1+c_3=0\\-c_1+c_2+c_3=0\\c_1+c_2+c_3=0\end{matrix}$

The only solution is $c_1=c_2=c_3=0$, thus the vectors are linearly independent.

5. (a) $W(\mathbf{x_1},\mathbf{x_2})=\begin{vmatrix}-2e^t & e^{-2t}\\ e^t & 0\end{vmatrix}=-e^{-t}\neq 0$ Thus the two solutions are linearly independent and form a fundamental set.

(b) $\mathbf{x}(t)=c_1e^t\begin{pmatrix}-2\\1\end{pmatrix}+c_2e^{-2t}\begin{pmatrix}1\\0\end{pmatrix}$

6. $$\begin{aligned}\frac{dQ_1}{dt}&=45+2\frac{Q_2}{20+2.5t}-5\frac{Q_1}{80}, \quad Q_1(0)=100\\ \frac{dQ_2}{dt}&=122.5+5\frac{Q_1}{80}-6\frac{Q_2}{20+2.5t}, \quad Q_2(0)=50\end{aligned}$$

7. $$\begin{aligned}x_1(t)&=6e^t-5e^{-2t}\\ x_2(t)&=-3e^t+4e^{-t}\\ x_3(t)&=-5e^{-2t}+4e^{-t}\end{aligned}$$

8. $\mathbf{x}(t)=-2e^t\begin{pmatrix}1\\0\end{pmatrix}+3e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-2e^t+3e^{-2t}\\3e^{-2t}\end{pmatrix}$

9. $$\begin{aligned}x(t)&=2e^{-t}+e^{5t}\\ y(t)&=-2e^{-t}+2e^{5t}\end{aligned}$$

10. The first eigenvalue/eigenvector pair gives the solution: $\mathbf{x_1}(t)=e^{-2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$

The second eigenvalue/eigenvector pair gives the two solutions:

$$\mathbf{x_2}(t)=e^t\begin{pmatrix}\cos t+\sin t\\2\cos t\\\cos t\end{pmatrix}, \quad \mathbf{x_3}(t)=e^t\begin{pmatrix}-\cos t+\sin t\\2\sin t\\\sin t\end{pmatrix}.$$

11. $$\begin{aligned}x(t)&=-4e^t\sin(2t)\\ y(t)&=4e^t\cos(2t)\end{aligned}$$