MAT 275 TEST 2 PRACTICE PROBLEMS

I. HODEs/IVP with constant coefficients.

1. Find a real valued solution to the following initial value problems. Sketch a graph of the solution.
   a. $y'' - 6y' + 13y = 0$, with $y(0) = 1,\ y'(0) = 1$.
   b. $y'' + 4y' + 4y = 0$, with $y(0) = 1,\ y'(0) = -4$.
   c. $6y'' + 7y' + 2y = 0$, with $y(0) = 7,\ y'(0) = -4$.
2. For which values of α (if any) are all solutions of $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$ unbounded as $t \to \infty$?
3. The characteristic equation of a homogeneous 9th order linear Differential Equation with constant coefficients has roots $r = 0$ with multiplicity three, $r = -2$ with multiplicity two, $r = -3 \pm 2i$ with multiplicity two. Write the general solution of the Differential Equation.
4. One solution of the DE $6y^{(4)} + 5y''' + 25y'' + 20y' + 4y = 0$ is $y = \cos(2x)$. Find the general solution.

II. Reduction of order:

1. The ODE $t^2y'' + 3ty' + y = 0$ has a solution $y_1(t) = \frac{1}{t}$ for $t > 0$. Find the general solution.
2. The ODE $2ty'' - 5y' + \frac{3}{t}y = 0$ has a solution $y_1(t) = t^3$ for $t > 0$. Find the general solution.

III. Undetermined coefficients

1. Find the general solution of the ODE $y'' + 2y' + y = e^{-t}$.
2. Solve the IVP: $y'' - y' - 2y = 6x + 6e^{-x},\ y(0) = 1,\ y'(0) = 0$.
3. Solve the IVP: $y'' - y' - 2y = 6te^{2t},\ y(0) = 0,\ y'(0) = 1$
4. Determine a suitable form for the particular solution $Y(t)$ if the method of undetermined coefficients is to be used. You do not need to determine the values of the coefficients.
   (i) $y'' + 3y' = 2t^2 + t^2e^{-3t} + \sin(3t)$
   (ii) $y'' + y = t(1 + \sin t)$
   (iii) $y'' - 5y' + 6y = e^t\cos(2t) + (3t + 4)e^{2t}\sin(t)$
   (iv) $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos(t) + 4t^2e^{-t}\sin(t)$
   (v) $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin(2t)$

IV. Mass-Spring system

1. Consider the IVP: $y'' + 4y = 0$ with $y(0) = -3$ and $y'(0) = 6$. Write the solution as $y(t) = R\cos(\omega_0 t - \delta)$.
2. A mass of 2 kilograms stretches a spring 0.5 meters. If the mass is set in motion from its equilibrium with a downward velocity of 10 cm/s, and there is no damping, write an IVP for the position u (in meters) of the mass at any time t (in seconds). Use $g = 9.8$ m/s^2 for the acceleration due to gravity.

3. For the following, choose the best description of the system from the following:
Simple Harmonic Motion (SHM) Overdamped (OD) Underdamped (UD) Critically Damped (CD) Beating (B) Resonant (R) Steady-State plus Transient (SST)

a. $y'' + 4y = 0$

b. $y'' + (1.8)^2 y = \cos(2t)$

c. $y'' + 4y = \cos(2t)$

d. $y'' + y' + y = 0$

e. $y'' + y' + y = \cos(t)$

f. $y'' + 2y' + y = 0$

4. The motion of a force mass-spring system is described by the following IVP:

$$u'' + 9u = \cos(3t), \qquad u(0) = 0, \qquad u'(0) = 0$$

(a) Explain why you expect resonance to occur.

(b) Solve this IVP and sketch the graph of the solution.

5. The motion of a force mass-spring system is described by the following IVP:

$$u'' + (2.8)^2 u = \cos(3t), \qquad u(0) = 0, \qquad u'(0) = 0$$

Explain why you expect the beats phenomenon to occur.

(a) Explain why you expect resonance to occur.

(b) Solve this IVP and write your solution in the form $A\sin(\alpha t)\sin(\beta t)$.

(c) Determine the length of the beats and the period of the oscillation.

6. A mass $m = 1$ is attached to a spring with constant $k = 2$ and damping constant γ. Determine the value of γ so that the motion is critically damped.

7. The position function of a mass-spring system satisfies the differential equation

$$mx'' + \gamma x' + kx = \cos(\omega t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

Assume $m = 1$ and $k = 9$.

If $\gamma \neq 0$, the amplitude of the forced oscillation is given by $C = \frac{1}{\sqrt{(9-\omega^2)^2 + \gamma^2\omega^2}}$

Assume $\gamma = 1$. Differentiate C to find the value of ω at which practical resonance occurs. Determine the corresponding value of C.

V. Laplace Transform

1. Use the definition of the Laplace transform to find $F(s) = \mathcal{L}\{f(t)\}$ for the following functions.

(a) $f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 0, & 4 \le t < \infty \end{cases}$

(b) $f(t) = \begin{cases} 0, & t < 2 \\ 6, & 2 \le t \end{cases}$

(c) $f(t) = \begin{cases} 0, & t < 2 \\ 5e^{-3t}, & 2 \le t \end{cases}$

(d) $f(t) = \begin{cases} 2e^{t}, & t < 1 \\ 2e, & 1 \le t \end{cases}$

2. Find the Laplace transform of the following functions.
   (a) $f(t) = \sin(2t)\cos(2t)$
   (b) $f(t) = 6e^{-2t}\sin(3t)$
   (c) $f(t) = 6te^{-2t}\sin(3t)$
   (d) $f(t) = t^2e^{3t}$

3. Find the inverse Laplace transform:
   (a) $F(s) = \frac{8}{s^2-s-6}$
   (b) $F(s) = \frac{4}{(s+2)(s^2+9)}$
   (c) $F(s) = \frac{s+4}{s^2+4s+29}$

4. The transform of the solution to a certain differential equation is given by $Y(s) = \frac{2s+7}{s^2+9}$. Determine the solution $y(t)$ of the differential equation.

5. Suppose that the function $y(t)$ satisfies the DE $y'' - 2y' - y = 3\sin(4t)$, with initial values $y(0) = -1$, $y'(0) = 1$. Find the Laplace transform of $y(t)$.

6. Consider the following IVP: $y'' + 6y' + 13y = 0,\ \ y(0) = 2,\ \ y'(0) = -1$.
   (a) Find the Laplace transform of the solution $y(t)$.
   (b) Find the solution $y(t)$ by inverting the transform.

7. Consider the following IVP: $y'' - 3y' - 10y = 5,\ \ y(0) = 2,\ \ y'(0) = -4$.
   (a) Find the Laplace transform of the solution $y(t)$.
   (b) Find the solution $y(t)$ by inverting the transform.

ANSWERS TO TEST 2 PRACTICE PROBLEMS

I.

1. (a) $y = -e^{-3t}\sin(2t) + e^{3t}\cos(2t)$ (b) $y(t) = e^{-2t} - 2te^{-2t}$ (c) $y(t) = 3e^{-2t/3} + 4e^{-t/2}$

2. The general solution is $y = c_1e^{\alpha t} + c_2e^{(\alpha-1)t}$. Thus all solutions are unbounded if $\alpha > 1$.

3. $c_1 + c_2t + c_3t^2 + c_4e^{-2t} + c_5te^{-2t} + e^{-3t}(c_6\cos(2t) + c_7\sin(2t)) + te^{-3t}(c_8\cos(2t) + c_9\sin(2t))$

4. Since $y = \cos(2x)$ is a solution, $2i$ and $-2i$ must be roots of the characteristic equation and $r^2 + 4$ must be a factor. Using long division, another factor is $6r^2 + 5r + 1$. Thus the characteristic equation can be written $(r^2 + 4)(3r + 1)(2r + 1)$ and the general solution is $y = c_1\cos(2x) + c_2\sin(2x) + c_3e^{-x/3} + c_4e^{-x/2}$.

II.

1. $y(t) = \frac{c_1}{t} + c_2\frac{\ln(t)}{t}$ **2.** $y(t) = c_1t^3 + c_2\sqrt{t}$

III.

1. $y = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}$ **2.** $y = \frac{3}{2}e^{2x} - 2e^{-x} + \frac{3}{2} - 3x - 2xe^{-x}$

3. $y = \frac{5}{9}e^{2t} - \frac{5}{9}e^{-t} + t^2e^{2t} - \frac{2}{3}te^{2t}$

4.

(i) $Y(t) = t(At^2 + Bt + C) + t(Dt^2 + Et + F)e^{-3t} + G\sin(3t) + H\cos(3t)$

(ii) $Y(t) = At + B + t(Ct + D)\sin(t) + t(Et + F)\cos(t)$

(iii) $Y(t) = Ae^t\cos(2t) + Be^t\sin(t) + (Ct + D)e^{2t}\cos(t) + (Et + F)e^{2t}\sin(t)$

(iv) $Y(t) = Ae^{-t} + t(Bt^2 + Ct + D)e^{-t}\cos(t) + t(Et^2 + Ft + G)e^{-t}\sin(t)$

(v) $Y(t) = At^2 + Bt + C + t^2(Dt + E)e^{2t} + (Ft + G)\cos(2t) + (Ht + I)\sin(2t)$

IV.

1. $y(t) = -3\cos(2t) + 3\sin(2t) = 3\sqrt{2}\cos\left(2t - \frac{3\pi}{4}\right)$

2. $2u'' + 39.2u = 0,\ u(0) = 0,\ u'(0) = 0.1$

3. a. undamped free motion: Simple Harmonic Motion

b. undamped motion with $\omega_0 = 1.8 \approx 2 = \omega$: Beats

c. undamped motion with$\omega_0 = 2 = \omega$: Resonance

d. free damped motion; roots of the characteristic equation are complex: Under Damped

e. damped motion with forcing term: Steady State plus Transient

f. free damped motion; roots of the characteristic equation are repeated: Critically Damped

4. (a) $\omega_0 = 3 = \omega$ (b) $u(t) = \frac{t}{6}\sin(3t)$

5. (a) $\omega_0 = 2.8 \approx 3 = \omega$ (b) $\frac{1}{1.16}(\cos(2.8t) - \cos(3t)) = \frac{2}{1.16}\sin(0.1t)\sin(2.9t)$

(c) Length of beats $= \frac{2\pi}{2(0.1)} = 10\pi$, Period of oscillation $= \frac{2\pi}{2.9}$.

6. $2\sqrt{2}$

7. $\omega = \sqrt{\frac{17}{2}} = \frac{\sqrt{34}}{2} \approx 2.91$. The corresponding maximum value of the amplitude is $C\left(\frac{\sqrt{34}}{2}\right) \approx 0.338$.

V.

1. (a) $\frac{3-3e^{-4s}}{s}$ (b) $\frac{6e^{-2s}}{s}$ (c) $\frac{5e^{-2(s+3)}}{s+3}$ (d) $\frac{2s-2e^{-(s-1)}}{(s-1)s}$

2. (a) $\frac{2}{s^2+16}$ (b) $\frac{18}{(s+2)^2+9}$ (c) $\frac{36(s+2)}{((s+2)^2+9)^2}$ (d) $\frac{2}{(s-3)^3}$

3. (a) $-\frac{8}{5}e^{-2t} + \frac{8}{5}e^{3t}$ (b) $\frac{4}{13}e^{-2t} - \frac{4}{13}\cos(3t) + \frac{8}{39}\sin(3t)$ (c) $e^{-2t}\cos(5t) + \frac{2}{5}e^{-2t}\sin(5t)$

4. $y(t) = 2\cos(3t) + \frac{7}{3}\sin(3t)$

5. $Y(s) = \frac{-s+3}{s^2-2s-1} + \frac{12}{(s^2-2s-1)(s^2+16)}$

6. (a) $Y(s) = \frac{2s+11}{s^2+6s+13}$ (b) $y(t) = 2e^{-3t}\cos(2t) + \frac{5}{2}e^{-3t}\sin(2t)$

7. (a) $Y(s) = \frac{2s-10}{s^2-3s-10} + \frac{5}{s(s^2-3s-10)}$ (b) $y(t) = \frac{33}{14}e^{-2t} + \frac{1}{7}e^{5t} - \frac{1}{2}$