MAT 275 TEST 1 PRACTICE

1. Determine if each of the following equations is separable (Yes or No), and /or linear (Yes or No). Record your answer in the following table. Do not attempt to solve the equations.

Equation	Separable	Linear
$y' = \frac{t+1}{yt}$		
$y' = \frac{yt}{t+1}$		
$y' = \cos(ty)$		
$y' - ty = t^3$		

2. a) Write down a first order linear ODE whose solutions all approach $y = 1$.

b) Write down a first order linear ODE such that solutions other than $y = -3$ all diverge from $y = -3$.

3. Consider the ODE $y' = y^2(y-1)(y+2)$

a) Determine all the equilibrium (constant) solutions and classify them as stable, unstable or semi-stable.

b) If $y(0) = -1$, what will be the behavior of the solution as $t \to \infty$?

c) If $y(0) = 0$, what will be the behavior of the solution as $t \to \infty$?

4. Which of the following differential equations best represents the slope field at the right?

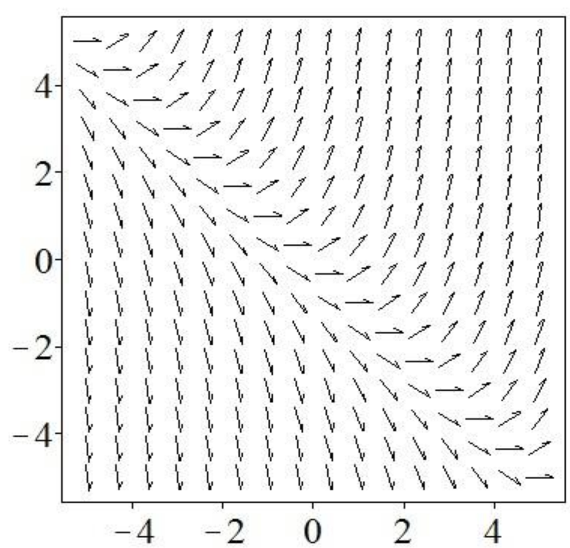
a) $y' = x + y$

b) $y' = x - y$

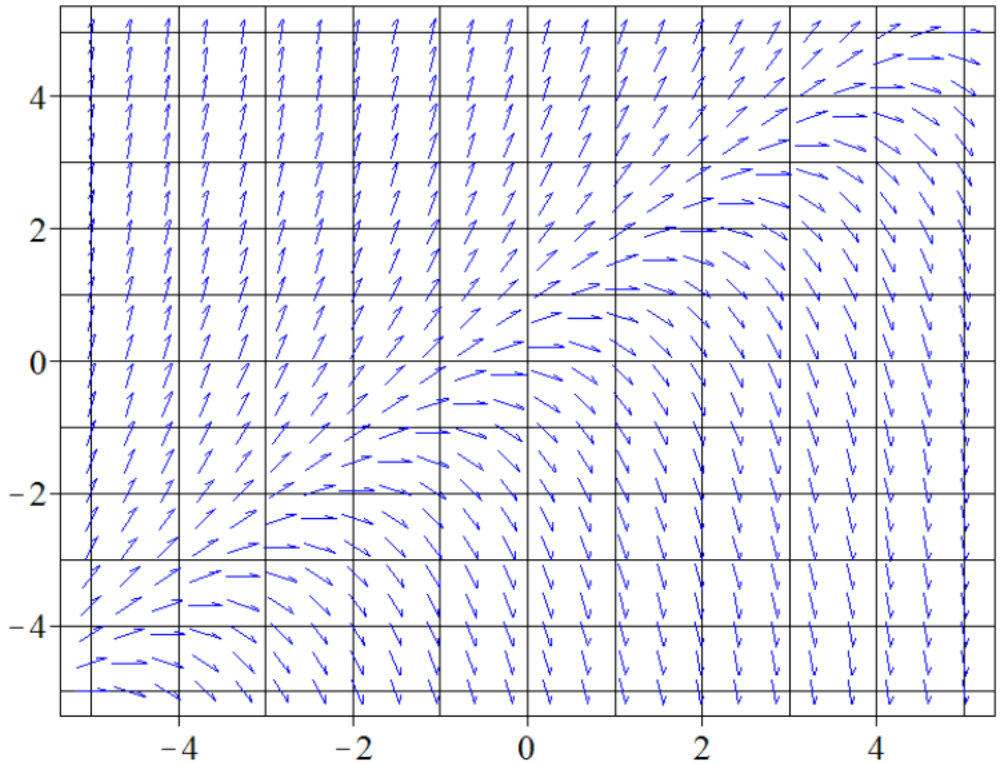
c) $y' = y - x$

d) $y' = xy$

e) $y' = \frac{x}{y}$



5. Consider the direction field below.



a) i. Sketch the graph of the solution which has initial value $y(0) = 1.5$.
ii. Sketch the solution which has initial value $y(0) = 0.5$.

b) Use your sketch to estimate the value of $y(1)$ for the solutions in (a) i. and (a) ii.

c) Some solutions of this differential equation grow when t is large and some do not. On the graph, sketch the curve representing the boundary between these two behaviors. Continue it in both directions until it leaves the direction field box. Label this curve "isocline".

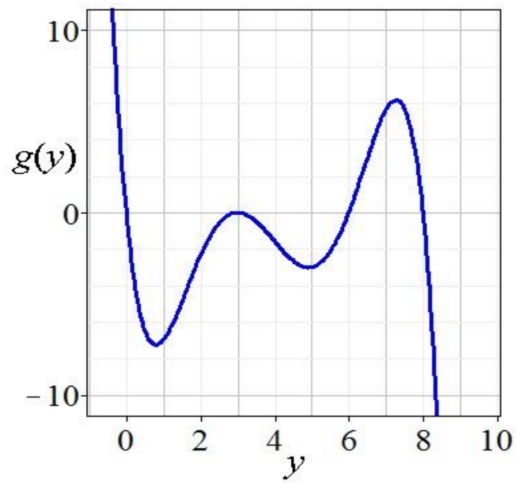
d) What happens to the two solutions in part (a) as $t \to \infty$?

e) Consider the solution with initial condition $y(0) = 1$. Estimate the value of $y(50)$. Explain your reasoning.

f) The direction field above comes from one of the following differential equations.
Tell which one it is and why.

I. $y' = (1 - y)y$
II. $y' = ye^{-t}$
III. $y' = y - t$

6. The population of Whooping Cranes has been found to obey the equation $\frac{dy}{dt} = g(y)$ (in appropriate units). Here's a graph of the function g(y) for y between -1 and 10.



a) Identify the equilibrium solutions and classify them as stable, unstable or semi-stable.

b) For what values of the initial condition at $t = 0$ will the population be extinct?
Give your answer in interval form.

7. Consider the differential equation: $x^2y'' - 3xy' + 5y = 2x^2\ln(x)$

(a) What is the order of the differential equation?

(b) Is the differential equation linear or nonlinear?

(c) Determine the value(s) of the constant A so that $y(x) = Ax^2\ln(x)$ is a solution to the differential equation.

8. Consider the differential equation: $-4x + y^2 + 2xyy' = 0$.

(a) What is the order of the differential equation?

(b) Is the Differential Equation linear or nonlinear?

(c) Determine the value(s) of the constant A so that $y(x) = A\sqrt{x}$ is a solution to the Differential Equation.

9. Determine the value(s) of the constant r so that $y = e^{rt}$ is a solution to $y'' - y' - 6y = 0$.

10. The general solution to a certain first order differential equation appears in implicit form $y^2 - \sin(x) = C$. Determine the value at $x = \frac{\pi}{2}$ of the solution satisfying $y(0) = -1$.

11. The general solution of the differential equation $xdy = ydx$ is a family of (determine the correct choice)

a) circles   b) parabolas   c) hyperbolas   d) lines passing through the origin

12. Use separation of variables to find the solution of the following Initial Value Problems.
Write your answer in <u>explicit</u> form and simplify as much as possible. For each IVP determine the interval in which the solution is defined.

a) $y' = \frac{2x}{y+1}$, $y(1) = -2$

b) $xy^2 + 3y^2 - x^2y' = 0$, $y(1) = 3$

c) $y' = \frac{2t}{t^2y+y}$, $y(1) = 2$

d) $\frac{x^2}{y^2-3}\frac{dy}{dx} = \frac{1}{2y}$, $y(1) = 2$

e) $\frac{dy}{dx} = 20yx^4$, $y(0) = 4$

13. A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min; thus the tank is empty after exactly 1 hour. Let $y(t)$ be the amount of salt in the tank after t minutes.

(a) Write an Initial Value Problem for the amount of salt in the tank at any time t (< 60).
(b) Solve the IVP in part (a) to find the amount of salt in the tank at any time t (< 60).
(c) Determine the amount of salt when the tank is half empty.

14. A completely filled 20 gallon tank originally contains 10 pounds of salt dissolved in water. Pure water enters the tank at the rate of 5 gallons/minute, and the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time t .

15. A ball with mass 0.2 kg is thrown upwards with initial velocity 26 meters per second. We assume that the forces acting on the body are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $c|v(t)|$ where $c = 0.1\frac{kg}{s}$ and $v(t)$ is the velocity of the ball at time t. The gravitational constant is $g = 9.8\frac{m}{s^2}$.

(a) Write and solve the differential equation for the velocity, $v(t)$.
(b) Find the formula for the position function at any time t, if the initial release is taken to be $s(0) = 0$.
(c) Find the time it takes for the ball to reach its maximum height, and the time at which the ball returns to its initial height.

16. A field mouse population satisfies the Initial Value Problem: $\frac{dp}{dt} = 0.5p - 400,\ p(0) = 600.$

(a) Find the time at which the population becomes extinct.
(b) Find the time at which the population becomes extinct if the initial condition is $p(0) = 800$.

17. Newton's law of cooling is $u' = -k(u - T)$ where $u(t)$ is the temperature of an object, t is in hours, T is a constant ambient temperature, and k is a positive constant. Suppose a building loses heat in accordance with Newton's law of cooling. Suppose that the rate constant k has the value $0.13hr^{-1}$. Assume that the interior temperature of the building is 76°F when the heating system fails and the external temperature is T=10°F.

(a) How long will it take for the interior temperature to fall to 32°F?
(b) What happens to the temperature $u(t)$ as $t \to \infty$?

18. The population of a city increases continuously at a rate proportional, at any time, to the population at that time. The population doubles in 50yr. Find the ratio of the population P to the initial population P_0 after 75 years.

19. Suppose P(t) denotes the size of an animal population at time t and its growth is described by the differential equation $\frac{dP}{dt} = 0.00P(1000 - P)$.Determine the value of P at which the population is growing fastest.

20. Solve the following Initial Value Problems using the method of integrating factor.

a) $y' = \frac{2}{t}y + 6t^4\ , \quad y(1) = -3$

b) $y' - 2y = 2e^{5t} + 5e^{2t}\ , \quad y(0) = -3$

c) $ty' = -\frac{\sin(t)}{t} - 2y\ , \quad y(\pi) = 1$

d) $(t+1)y' - 2y = 2t\,, \qquad y(0) = 4$

e) $\frac{dy}{dt} + 0.2ty = 5t\,, \qquad y(0) = 6$

21. Determine (without solving the problem) the maximal interval in which the solution of the given initial value problem is guaranteed to exist: $ty' + \tan(t)\,y = \sin(t)\,,\ \ y(\pi) = 6$.

22. a) Verify that both $y_1 = 2t - 1$ and $y_2 = t^2$ are solutions to $y' = 2\left(t - \sqrt{t^2 - y}\right)$.
In which intervals in *t* are the solutions valid?

b) Does the existence of two solutions of the given problem contradict any known theorem about existence and uniqueness of solutions to Differential Equations?

23. Consider the following differential equations. Determine if the Existence and Uniqueness Theorem does or does not guarantee existence and uniqueness of a solution of each of the following initial value problems.

I. $\frac{dy}{dx} = \sqrt{x-y},\ \ y(2) = 2$

II. $\frac{dy}{dx} = \sqrt{x-y},\ \ y(2) = 1$

III. $y\frac{dy}{dx} = x - 1,\ \ y(0) = 1$

IV. $y\frac{dy}{dx} = x - 1,\ \ y(1) = 0$

24. Use Euler's method and two steps with $\Delta x = 0.1$ for the differential equation *y*' = *y*, with initial value $y(0) = 1$, to find the approximate value of $y(0.2)$.

25. Use Euler's method and three steps with $\Delta t = 0.2$ for the differential equation $y' = 3t - 2y^2 - 2$, with initial value $y(1) = 1$, to find the approximate value of $y(1.6)$.

26. Which statement about Euler's method is false?

I. If you halve the step size, you approximately halve the error.

II. Euler's method never gives exact solutions.

III. Euler's method assumes that the slope of a solution curve is the same at all points in a short interval.

IV. Often, when applying Euler's method, the more steps you take the smaller the error.

V. Euler's method is used to string together a set of linearizations that approximate the curve.

27. Find a real valued solution to the following initial value problems. Sketch a graph of the solution.

a) $y'' + 3y' + 2y = 0,$ with $y(0) = 3,\ y'(0) = 0$

b) $2y'' + 3y' - 2y = 0,$ with $y(0) = 1,\ y'(0) = 7$

c) $y'' - 5y = 0,$ with $y(0) = 1,\ y'(0) = -1$

d) $3y'' - 4y' = 0,$ with $y(0) = 2,\ y'(0) = -8$

28. Determine the longest interval on which the given initial value problem is certain to have a unique twice differentiable solution.

(a) $(x-3)y'' + \frac{x}{x-3}y' + \sqrt{x-1}\,y = 0, \quad y(2) = 0,\ y'(2) = 1$

(b) $(t-1)y'' + ty' + y = \sec(t), \quad y(0) = 1,\ y'(0) = 3$

(c) $t(t-4)y'' + 3y' + \ln(t)\,y = \sin(t), \quad y(1) = 1,\ y'(1) = 1$

29. Which of the following is a true statement?

I. Two functions defined on an open interval I are said to be linearly independent on I provided that one is a constant multiple of the other on I.

II. Two functions defined on an open interval I are said to be linearly dependent on I provided that one is a constant multiple of the other on I.

30. Which of the following pairs of functions is linearly independent on the entire real line?

A. $\{\sin(x),\ \cos(x)\}$ B. $\{e^x,\ xe^x\}$ C. $\left\{x,\ \left(\frac{e}{\pi}\right)^3 x\right\}$ D. $\{x,\ 3x\}$

E. $\{1,\ e^{-t}\}$ F. $\{\cos(t),\ \sin(t+\pi/2)\}$ G. $\{e^{-2t}\cos(2t),\ e^{-2t}\sin(2t)\}$

H. $\{2e^{-t},\ 4e^{-t+3}\}$ I. $\{e^{2t},\ e^{2t}-6\}$ J. $\{x,\ |x|\}$

31. Which of the following is NOT a fundamental set of solutions for $y'' - y = 0$?

A. $\{e^t,\ e^{-t}\}$ B. $\{2e^t,\ 2e^{-t}\}$ C. $\{te^t,\ e^{-t}\}$ D. $\left\{(e^t+e^{-t}),\ \frac{1}{2}(e^t+e^{-t})\right\}$

E. $\left\{\frac{1}{2}(e^t+e^{-t}),\ \frac{1}{2}(e^t-e^{-t})\right\}$ F. $\left\{\frac{1}{2}(e^t+e^{-t}),\ e^t\right\}$

32. Suppose $y_1(t) = t$ and $y_2(t) = t^2$ are both solutions of the second order linear equation $y'' + p(t)y' + q(t) = 0$. Which of the functions below are guaranteed to also be solutions of the same equation?

A. $y = t^2 - 1$ B. $y = 5t$ C. $y = -9t^2 + 17t$ D. $y = 0$

33. Consider the ODE $t^2y'' + 3ty' + y = 0$ with the initial conditions $y(1) = 1,\ y'(1) = 1$.

(i) What is the maximum interval of validity, I, of the solution?

(ii) Verify that the functions $y_1(t) = t^{-1}$ and $y_2(t) = t^{-1}\ln(t)$ satisfy the ODE for t in the interval I.

(iii) Find the Wronskian $W(y_1, y_2)$ to show that $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

(iv) Solve the initial value problem.

Answers:

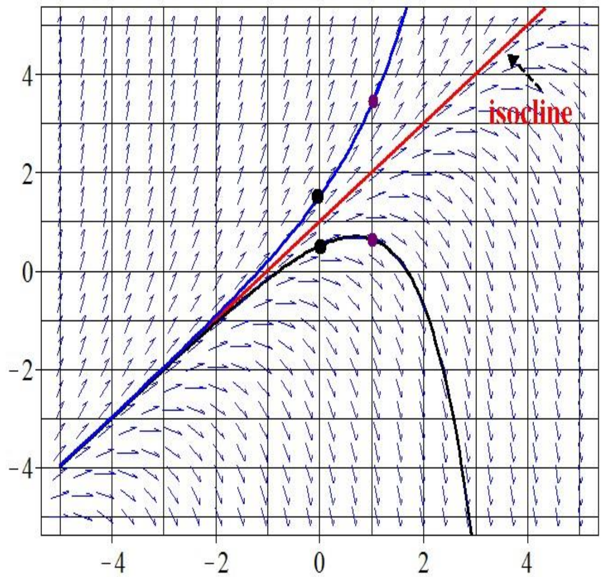
1. 1st DE: Yes, No ; 2nd DE: Yes, Yes; 3rd DE: No, No; 4th DE: No, Yes

2. Possible answers: (a) $y' = 1 - y$ (b) $y' = y + 3$

3. (a) $y = -2$ stable, $y = 0$ semi-stable, $y = 1$ unstable

(b) $\lim_{t\to\infty} y(t) = -2$ (c) $\lim_{t\to\infty} y(t) = 0$

4. (a)



5. (b) i. $y(1) \approx 3.3$ ii. $y(1) \approx 0.6$

(d) The first solution approaches ∞ ;

The second solution approaches $-\infty$

(e) 51 (f) III.

6. (a) $y = 0$ stable; $y = 3$ semi-stable; $y = 6$ unstable; $y = 8$ stable (b) [0,3)

7. (a) second order (b) linear (c) $A = 2$

8. (a) first (b) non linear (c) $A = \pm\sqrt{2}$

9. $r = 3$, $r = -2$ **10.** $= -\sqrt{2}$ **11.** d)

12.

a) $y = -1 - \sqrt{2x^2 - 1}$ Interval: $\left(\frac{1}{\sqrt{2}}, \infty\right)$

b) $y = \frac{-1}{\ln(x) - \frac{3}{x} + \frac{8}{3}} = \frac{-3x}{3x\ln(x) - 9 + 8x}$ Interval: $(0, 1.089)$

c) $y = -\sqrt{2\ln(t^2 + 1) + 4 - 2\ln(2)}$ Interval: $(-\infty, \infty)$

d) $y = \sqrt{3 + e^{1 - \frac{1}{x}}}$ Interval: $(0, \infty)$

e) $y = 4e^{4x^5}$ Interval: $(-\infty, \infty)$

13. (a) $\frac{dy}{dt} = 2 - \frac{3y}{60 - t}$, $y(0) = 0$

(b) $y(t) = 60 - t - \frac{(60 - t)^3}{3600}$

(c) $y(30) = 22.5$ lbs

14. $y(t) = 10e^{-t/4}$

15. (a) $0.2\frac{dv}{dt} = -0.1v - (0.2)(9.8)$ or $\frac{dv}{dt} = -0.5v - 9.8$; $v(t) = 45.6e^{-0.5t} - 19.6$

(b) $s(t) = 91.2 - 91.2e^{-0.5t} - 19.6t$

(c) $v = 0$ when $t = -2\ln\left(\frac{19.6}{45.6}\right) \cong 1.69$ sec. $s = 0$ when $t \cong 4.03$ sec.

16. (a) $t = 4\ln(2) \approx 2.77$ (b) The population will never become extinct.

17. (a) 8.45 hrs (b) $\lim_{t\to\infty} u(t) = 10$ ($u(t)$ will approach the external temperature).

18. $2\sqrt{2}$

19. $P = 500$

20.

(a) $y(t) = t^2(2t^3 - 5)$ (b) $y(t) = \left(\frac{2}{3}e^{3t} + 5t - \frac{11}{3}\right)e^{2t}$

(c) $y(t) = \frac{\cos(t)+\pi^2+1}{t^2}$ (d) $y(t) = -2t - 1 + 5(t+1)^2$

(e) $y(t) = 25 - 19e^{-t^2/10}$

21. $\frac{\pi}{2} < t < \frac{3\pi}{2}$

22. (a) $y_1(t)$ is a solution for $t \geq 1$; $y_2(t)$ is a solution for all t;

(b) f_y is not continuous at (1, 1).

23. Only II and III are guaranteed to have a unique solution.

24. $y(0.2) \approx 1.210$

25. $y(1.6) \approx 1.005$

26. II

27. (a) $y(t) = -3e^{-2t} + 6e^{-t}$ (b) $y(t) = 4e^{t/2} - 3e^{-2t}$

(c) $y(t) = \frac{5-\sqrt{5}}{10}e^{\sqrt{5}t} + \frac{5+\sqrt{5}}{10}e^{-\sqrt{5}t}$ (d) $y(t) = 8 + 6e^{4t/3}$

28. (a) $1 < x < 3$ (b) $-\frac{\pi}{2} < t < 1$ (c) $0 < t < 4$

29. II

30. A, B, E, G, I, J

31. C: te^t is not a solution; D: the two functions are not linearly independent

32. By the principle of superposition, B, C, D

33. (i) $t > 0$

(iii) $W(y_1, y_2) = \frac{1}{t^3}$. Since the Wronskian is nonzero on I, $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions.

(iv) $y = \frac{1+2\ln(t)}{t}$