

LIST OF CONCEPTS AND SKILLS FOR CHAPTER 6-7

New material covers sections 6.3-6.5, 7.1-7.6, 7.8 from *MAT 275-Elementary Differential Equations*, by Boyce and DePrima - WileyPLUS, 10th edition. (Material from earlier sections are on previous reviews.)

CHAPTER 6: THE LAPLACE TRANSFORM

6.3: Step Functions

- Know the definition of the unit step function (Heaviside function), $u_c(t) = \text{step}(t - c)$, and how to write a piecewise function in terms of the unit step functions and use the appropriate entry in the table to find the Laplace transform.
- Know how to graph functions involving the unit step function.
- Know how to apply Theorem 6.3.1: $\mathcal{L}\{u_c(t)f(t - c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs}F(s)$, and conversely, $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t - c)$, where $F(s) = \mathcal{L}\{f(t)\}$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$.
- Know how to use the translation property: $\mathcal{L}\{e^{ct}f(t)\} = F(s - c)$.

6.4: Differential Equations with Discontinuous Forcing Equations.

Know how to solve Differential Equations where the forcing term is given by a piecewise continuous function. In these cases the function needs to be written in terms of unit step functions $u_c(t)$ in order to evaluate the Laplace.

6.5: Impulse Functions

Know the definition of the Dirac delta function, $\delta(t - t_0)$, and know how to solve differential equations where the forcing terms involves delta functions.

Some Laplace transform formulas:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$$

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$$

$$\mathcal{L}\{e^{at} \cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at} \sin(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs}$$

$$\mathcal{L}\{u_c(t)f(t - c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t + c)\}$$

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

CHAPTER 7: LINEAR SYSTEMS

Section 7.1: Introduction to linear systems

- Know how to transform a linear ODE of any order n into a system of n first-order ODEs.
- Know how to write a system of first-order linear ODEs in matrix form.
- Know how to solve simple two-dimensional systems by rewriting them as a single higher order ODE.
- Know how to determine the trajectories of the phase portraits for simple 2-dimensional systems.
- Set up the system of differential equations and initial conditions that model the mixing of salt water with two interconnected tanks.

Section 7.3: Linear Independence, Eigenvalues and Eigenvectors

- Know the definition of linearly independent and linearly dependent vectors.
- Know how to find eigenvalues and eigenvectors of a matrix.

Section 7.4: Basic Theory of Systems of First Order Linear Equations

- Know how to write a linear system in the form $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$ with $P(t)$ the coefficient matrix and \mathbf{x} and $\mathbf{f}(t)$ the appropriate vectors.
- Know how to verify (by substituting) that a given vector function is a solution to a given system.
- Given an $n \times n$ homogeneous linear system of differential equations written in matrix form as $\mathbf{x}' = P(t)\mathbf{x}$ with $P(t)$ a matrix whose entries are continuous, know how to find the Wronskian of n solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t)$. If the Wronskian of the n solutions $W(\mathbf{x}_1(t), \mathbf{x}_2(t), \dots, \mathbf{x}_n(t))$ is nonzero at every point of some interval I , then the solutions are linearly independent on I .
- Given the general solution, know how to use the Initial Conditions to determine the value of the constants c_1, c_2, \dots, c_n .

Section 7.5: Homogeneous Linear Systems with Constant coefficients

Consider a system of n linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the coefficient matrix \mathbf{A} is real valued.

If all the eigenvalues of \mathbf{A} are real and distinct, then there are n linearly independent eigenvectors and the general solution of the system is given by $\mathbf{x} = c_1\mathbf{v}_1e^{r_1t} + c_2\mathbf{v}_2e^{r_2t} + \dots + c_n\mathbf{v}_ne^{r_nt}$ where r_1, r_2, \dots, r_n are the eigenvalues and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are the corresponding n linearly independent eigenvectors.

Section 7.6: Complex Eigenvalues

Consider a system of n linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the coefficient matrix \mathbf{A} is real valued.

If the eigenvalues are complex, then they occur in conjugate pairs and so do the associated eigenvectors. Let $r = p + qi$ be a complex eigenvalue and $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ be the associated eigenvector, then, using Euler's formula, the solution can be written as

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{v}e^{rt} = (\mathbf{a} + i\mathbf{b})e^{pt}(\cos(qt) + i\sin(qt)) \\ &= e^{pt}(\mathbf{a}\cos(qt) - \mathbf{b}\sin(qt)) + ie^{pt}(\mathbf{a}\sin(qt) + \mathbf{b}\cos(qt))\end{aligned}$$

Taking the real and imaginary part of $\mathbf{x}(t)$ we find the **two real valued** solutions of the system:

$$\mathbf{x}_1(t) = e^{pt}(\mathbf{a} \cos(qt) - \mathbf{b} \sin(qt)) \text{ and } \mathbf{x}_2(t) = e^{pt}(\mathbf{a} \sin(qt) + \mathbf{b} \cos(qt))$$

It can be shown that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are linearly independent and therefore they can be used to write the general solution $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.

NOTE: Rather than memorizing the formulas for $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, you should be able to derive them for every specific example using Euler's formula and separating the real and imaginary parts.

Section 7.8: Repeated Eigenvalues

Consider a system of two linear homogeneous equations with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where the coefficient matrix \mathbf{A} is real valued.

If the eigenvalues are repeated with eigenvalue r (real and multiplicity two), then for any eigenvector, \mathbf{v}_1 , for r , we have that $\mathbf{x}_1(t) = \mathbf{v}_1 e^{rt}$ is a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

A second linearly independent solution is of the form $\mathbf{x}_2(t) = (\mathbf{v}_2 + t\mathbf{v}_1)e^{rt}$ where $(\mathbf{A} - r\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$. (the vector \mathbf{v}_2 is not be unique). The general solution is $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$.