MAT 275 CHAPTER 6, 7 PRACTICE PROBLEMS

(Material from earlier sections are on previous reviews)

6.3. Step Functions

1. Find the Laplace transform of the following functions.

(a) $f(t) = (t+3)u_7(t)$

(b) $f(t) = t^2 u_3(t)$

(c) $f(t) = \begin{cases} 1, & 0 \le t < 2 \\ t^2 - 4t + 4, & t \ge 2 \end{cases}$

(d) $f(t) = \begin{cases} t, & 0 \le t < 3 \\ 5, & t \ge 3 \end{cases}$

(e) $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

(f) $f(t) = \begin{cases} \cos(\pi t), & t < 4 \\ 0, & t \ge 4 \end{cases}$

(g) $f(t) = \begin{cases} t, & 0 \le t < 1 \\ e^t, & t \ge 1 \end{cases}$

2. Find the inverse Laplace transform:

(a) $F(s) = \frac{e^{-3s}}{s-2}$

(b) $F(s) = \frac{1+e^{-2s}}{s^2+6}$

(c) $F(s) = \frac{3}{s} + \frac{4}{s^2} + \frac{5s}{s^2+9} - e^{-3s}\left(\frac{3}{s} + \frac{4}{s^2} + \frac{5s}{s^2+9}\right)$

6.4. Solutions of IVP with Discontinuous Forcing Functions

3. Suppose that the function $y(t)$ satisfies the DE $y'' - 2y' - 8y = f(t)$, with $f(t) = \begin{cases} \sin(\pi t), & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$ and initial values $y(0) = -1,\ y'(0) = 3$. Find the Laplace transform of $y(t)$.

4. Consider the following IVP: $y'' + 16y = 2 - 2u_3(t),\ \ y(0) = 0,\ y'(0) = 0.$

(a) Find the Laplace transform of the solution $y(t)$.

(b) Find the solution $y(t)$ by inverting the transform.

6.5. Impulse Functions

5. A mass $m = 1$ is attached to a spring with constant $k = 5$ and damping constant $c = 2$. At the instant $t = \pi$, the mass is struck with a hammer, providing an impulse $p = 10$. Also, $x(0) = 0$ and $x'(0) = 0$.

a) Write the differential equation governing the motion of the mass.

b) Find the Laplace transform of the solution $x(t)$.

c) Apply the inverse Laplace transform to find the solution.

6. Consider the following IVP: $y'' + 4y = 5\delta(t-3),\ \ y(0) = 1,\ y'(0) = 2.$

(a) Find the Laplace transform of the solution $y(t)$.

(b) Find the solution $y(t)$ by inverting the transform.

7.1. Introduction to Systems

7. Transform the given IVP into an initial value problem for two first order equations.

(a) $y'' - 6y' + 8y = 0$

(b) $u'' + 4u' + 5tu = 7 - \sin(2t)$

8. Write the following IVP for a system of two linear ODEs as an IVP for a single second-order ODE.

(a) $\begin{matrix} x' = -y & x(0) = 1 \\ y' = 10x - 7y & y(0) = -7 \end{matrix}$

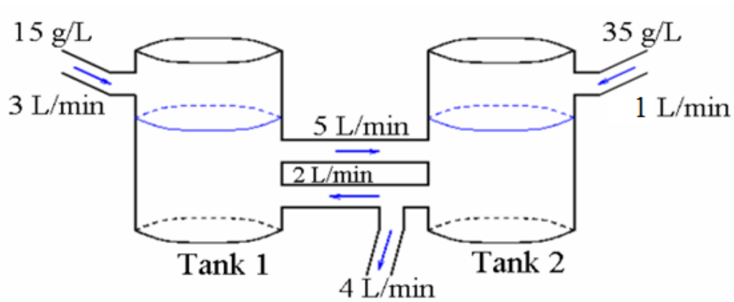
(b) Solve the above IVP

9. Match the description of the phase portrait with the corresponding system (one description will not match).

I $x' = y,\ y' = -x$ II $x' = y, y' = x$ III $x' = -2y,\ y' = x$

A. circles   B. ellipses   C. hyperbolas   D. parallel lines

10. Consider two interconnecting tanks as shown in the figure. Tank 1 initially contains 80 L (liters) of water and 100 g (grams) of salt, while Tank 2 initially contains 65 L of water and 50 g of salt. Water containing 15g/L of salt is poured into tank 1 at a rate of 3 L/m while the mixture flowing into tank 2 contains a salt concentration of 35 g/L and is flowing at a rate of 1 L/min. The mixture flows from tank 1 to tank 2 at a rate of 5 L/min. The mixture drains from tank 2 at a rate of 6 L/min, of which some flows back into Tank 1 at a rate of 2 L/min, while the remainder leaves the tank. Let Q1 and Q2, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process.



7.2-7.4. Matrices, Basic Theory of Systems

11. Verify that $\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix}e^t + \begin{pmatrix}2\\2\end{pmatrix}te^t$ is a solution of the system $\mathbf{x}' = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\mathbf{x} + \begin{pmatrix}1\\-1\end{pmatrix}e^t$.

12. Given the system $x' = tx - y + e^t z,\ \ y' = 2x + t^2 y - z,\ \ z' = e^{-t} + 3ty + t^3 z,$ define $\mathbf{x}$, $P(t)$ and $\mathbf{f}(t)$ such that the system is represented as $\mathbf{x}' = P(t)\mathbf{x} + \mathbf{f}(t)$.

13. Consider the second order initial value problem $u'' + 2u' + 2u = 3\sin(t)$, $u(0) = 2, u'(0) = -1$. Change the IVP into a first order initial value system and write the resulting system in matrix form.

14. Are the vectors $\mathbf{x_1} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\mathbf{x_2} = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $\mathbf{x_3} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ linearly independent?

15. Consider the system $\mathbf{x}' = \begin{pmatrix}-2 & -6\\0 & 1\end{pmatrix}\mathbf{x}$. Two solutions are $\mathbf{x_1} = \begin{pmatrix}-2\\1\end{pmatrix}e^t$ and $\mathbf{x_2} = \begin{pmatrix}1\\0\end{pmatrix}e^{-2t}$.

(a) Use the Wronskian to verify that the two solutions are linearly independent.

(b) Write the general solution of the system.

16. Suppose the system $\mathbf{x}' = A\mathbf{x}$ has general solution $\mathbf{x}(t) = c_1\begin{pmatrix}-2\\1\\0\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix}e^{-2t} + c_3\begin{pmatrix}0\\1\\1\end{pmatrix}e^{-t}$, where $\mathbf{x}(t) = \begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{pmatrix}$. Given the initial condition $\mathbf{x}(t) = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, find $x_1(t),\ x_2(t),$ and $x_3(t)$.

7.5. Homogeneous Linear Systems with Constant Coefficients; Real, Distinct Eigenvalues

17. Solve the IVP $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix}1 & -3\\0 & -2\end{pmatrix}$ and $\boldsymbol{x}(0) = \begin{pmatrix}1\\3\end{pmatrix}$.

18. Solve the IVP $\begin{matrix}x' = x + 2y\\ y' = 4x + 3y\end{matrix}$ with $x(0) = 3,\ y(0) = 0$.

7.6. Homogeneous Linear Systems with Constant Coefficients; Complex Eigenvalues

19. Find the general solution to $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix}1 & -2\\4 & -3\end{pmatrix}$.

20. Solve the IVP $\begin{matrix}x' = x + 2y\\ y' = -5x - y\end{matrix}$ with $x(0) = 4,\ y(0) = 1$.

21. Suppose A is real 3×3 matrix that has the following eigenvalues and eigenvectors:
$-2, \begin{pmatrix}1\\1\\1\end{pmatrix},\ 1+i, \begin{pmatrix}1-i\\2\\1\end{pmatrix},\ 1-i, \begin{pmatrix}1+i\\2\\1\end{pmatrix}$. Find a fundamental set of real valued solutions to $\mathbf{x}' = A\mathbf{x}$.

7.8. Homogeneous Linear Systems with Constant Coefficients; Repeated Eigenvalues

22. Find the general solution to $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{pmatrix}-5 & 9\\-1 & 1\end{pmatrix}$.

23. Solve the IVP $\begin{matrix}x' = 4x + 3y\\ y' = -3x - 2y\end{matrix}$ with $x(0) = 1,\ y(0) = -2$.

ANSWERS TO CHAPTER 6-7 PRACTICE PROBLEMS

6.3. Step Functions

1. (a) $\mathcal{L}\{f(t)\} = e^{-7s}\,\mathcal{L}\{t + 10\} = e^{-7s}\left(\frac{1}{s^2} + \frac{10}{s}\right)$

(b) $\mathcal{L}\{f(t)\} = e^{-3s}\mathcal{L}\{(t+3)^2\} = e^{-3s}\mathcal{L}\{t^2 + 6t + 9\} = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$

(c) $f(t) = 1 + u_2(t)(t^2 - 4t + 3)$ so $\mathcal{L}\{f(t)\} = \frac{1}{s} + e^{-2s}\mathcal{L}\{(t+2)^2 - 4(t+2) + 3\}$
$= \frac{1}{s} + e^{-2s}\mathcal{L}\{t^2 - 1\} = \frac{1}{s} + e^{-2s}\left(\frac{2}{s^3} - \frac{1}{s}\right)$

(d) $f(t) = t - u_3(t)(t-5)$ so $\mathcal{L}\{f(t)\} = \frac{1}{s^2} - e^{-3s}\mathcal{L}\{t + 3 - 5\} = \frac{1}{s^2} + e^{-3s}\left(\frac{1}{s^2} - \frac{2}{s}\right)$

(e) $f(t) = u_\pi(t)(t-\pi) - u_{2\pi}(t)(t-\pi)$ so $\mathcal{L}\{f(t)\} = e^{-\pi s}\mathcal{L}\{(t+\pi) - \pi\}$
$- e^{-2\pi}\mathcal{L}\{(t+2\pi) - \pi\} = e^{-\pi s}\mathcal{L}\{t\} - e^{-2\pi}\mathcal{L}\{t+\pi\} = \frac{e^{-\pi s}}{s^2} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right)$

(f) $f(t) = \cos(\pi t) - u_4(t)\cos(\pi t)$ so $\mathcal{L}\{f(t)\} = \frac{s}{s^2+\pi^2} - e^{-4s}\mathcal{L}\{\cos(\pi(t+4))\}$
$= \frac{s}{s^2+\pi^2} - e^{-4s}\mathcal{L}\{\cos(\pi t)\cos(4\pi) - \sin(\pi t)\sin(4\pi)\}$
$= \frac{s}{s^2+\pi^2} - e^{-4s}\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2} - \frac{se^{-4s}}{s^2+\pi^2}$

(g) $f(t) = t + u_1(t)(e^t - t)$ so $\mathcal{L}\{f(t)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{e^{t+1} - (t+1)\}$

$= \frac{1}{s^2} + e^{-s}\left(\frac{e}{s-1} - \frac{1}{s^2} - \frac{1}{s}\right)$

2. (a) The inverse Laplace transform is $u_3(t)f(t-3)$ where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

Thus $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s-2}\right\} = u_3(t)e^{2(t-3)}$.

(b) $F(s) = \frac{1}{\sqrt{6}}\frac{\sqrt{6}}{s^2+6} + \frac{e^{-2s}}{\sqrt{6}}\frac{\sqrt{6}}{s^2+6}$, thus $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{\sqrt{6}}\sin(\sqrt{6}\,t) + \frac{1}{\sqrt{6}}u_2(t)\sin\left(\sqrt{6}(t-2)\right)$

(c) $\mathcal{L}^{-1}\{F(s)\} = 3 + 4t + 5\cos(3t) - u_3(t)(3 + 4(t-3) + 5\cos(3(t-3)))$

6.4. Solutions of IVP with Discontinuous Forcing Functions

3. $Y(s) = \frac{-s+3}{s^2-2s-8} + \frac{\pi}{(s^2-2s-8)(s^2+\pi^2)} + e^{-s}\frac{\pi}{(s^2-2s-8)(s^2+\pi^2)}$

4. (a) $Y(s) = \frac{2}{s(s^2+16)} - e^{-3s}\frac{2}{s(s^2+16)} = \frac{1}{8}\left(\frac{1}{s}\right) - \frac{1}{8}\left(\frac{s}{s^2+16}\right) - e^{-3s}\left(\frac{1}{8}\left(\frac{1}{s}\right) - \frac{1}{8}\left(\frac{s}{s^2+16}\right)\right)$

(b) $y(t) = \frac{1}{8} - \frac{1}{8}\cos(4t) + u_3(t)\left(\frac{1}{8} - \frac{1}{8}\cos(4(t-3))\right)$

6.5. Impulse Functions

5. (a) $x'' + 2x' + 5x = 10\delta(t-\pi)$ (b) $X(s) = \frac{10e^{-\pi s}}{(s+1)^2+4}$

(c) $x(t) = 5u_\pi(t)e^{-(t-\pi)}\sin(2(t-\pi)) = 5u_\pi(t)e^{-(t-\pi)}\sin(2t)$

6. (a) $Y(s) = \frac{s+2}{s^2+4} + 5e^{-3s}\frac{1}{s^2+4}$ (b) $y(t) = \cos(2t) + \sin(2t) + \frac{5}{2}u_3(t)\sin(2(t-3))$

7.1. Introduction to Systems

7. (a) $x_1' = x_2,\ x_2' = -8x_1 + 6x_2$ (b) $x_1' = x_2,\ x_2' = -5tx_1 - 4x_2 + 7 - \sin(2t)$

8. (a) $y'' + 7y' + 10y = 0,\ y(0) = -7,\ y'(0) = 59$

(b) $x(t) = 4e^{-2t} - 3e^{-5t},\ \ y(t) = 8e^{-2t} - 15e^{-5t}$

9. I: Solving $\frac{dy}{dx} = -\frac{x}{y}$ yields $x^2 + y^2 = C$, hence the trajectories are circles and I matches A.

II: Solving $\frac{dy}{dx} = \frac{x}{y}$ yields $y^2 - x^2 = C$, hence the trajectories are hyperbolas and II matches C.

III: Solving $\frac{dy}{dx} = -\frac{x}{2y}$ yields $\frac{x^2}{2} + y^2 = C$, hence the trajectories are ellipses and III matches B

10. The system IVP is

$$\frac{dQ_1}{dt} = 45 + 2\frac{Q_2}{65} - 5\frac{Q_1}{80} \qquad Q_1(0) = 100$$

$$\frac{dQ_2}{dt} = 35 + 5\frac{Q_1}{80} - 6\frac{Q_2}{65} \qquad Q_2(0) = 50$$

(Note: The equilibrium solution is $(1360, 1300)$.)

7.2-7.4. Matrices, Basic Theory of Systems

11. Differentiating the given $\mathbf{x}$ yields $\mathbf{x}' = \begin{pmatrix}1\\0\end{pmatrix}e^t + \begin{pmatrix}2\\2\end{pmatrix}(e^t + te^t) = \begin{pmatrix}3e^t + 2te^t\\2e^t + 2te^t\end{pmatrix}$

Substituting **x** into the right hand side of the DE yields:

$$\begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}\begin{pmatrix}e^t + 2te^t\\ 2te^t\end{pmatrix} + \begin{pmatrix}e^t\\ -e^t\end{pmatrix} = \begin{pmatrix}2e^t + 4te^t - 2te^t\\ 3e^t + 6te^t - 4te^t\end{pmatrix} + \begin{pmatrix}e^t\\ -e^t\end{pmatrix} = \begin{pmatrix}3e^t + 2te^t\\ 2e^t + 2te^t\end{pmatrix} = \mathbf{x}'$$

12. $\mathbf{x} = \begin{pmatrix}x\\ y\\ z\end{pmatrix}, \quad P(t) = \begin{pmatrix}t & -1 & e^t\\ 2 & t^2 & -1\\ 0 & 3t & t^3\end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix}0\\ 0\\ e^{-t}\end{pmatrix}$

13. $\begin{pmatrix}u'\\ v'\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2 & -2\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}0\\ 3\sin(t)\end{pmatrix}, \quad \begin{pmatrix}u(0)\\ v(0)\end{pmatrix} = \begin{pmatrix}2\\ -1\end{pmatrix}$

14. yes, determinant of the column vectors is 0.

15. (a) $W(\mathbf{x_1}, \mathbf{x_2}) = \begin{vmatrix}-2e^t & e^{-2t}\\ e^t & 0\end{vmatrix} = e^{-t} \neq 0$. Thus the two solutions are linearly independent and form a fundamental set.

(b) $\mathbf{x}(t) = c_1\begin{pmatrix}-2\\ 1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\ 0\end{pmatrix}e^{-2t}$.

16.
$x_1(t) = 6e^t - 5e^{-2t}$
$x_2(t) = -3e^t + 4e^{-t}$
$x_3(t) = -5e^{-2t} + 4e^{-t}$

7.5. Homogeneous Linear Systems with Constant Coefficients; Real, Distinct Eigenvalues

17. $\mathbf{x}(t) = -2\begin{pmatrix}1\\ 0\end{pmatrix}e^t + 3\begin{pmatrix}1\\ 1\end{pmatrix}e^{-2t} = \begin{pmatrix}-2e^t + 3e^{-2t}\\ 3e^{-2t}\end{pmatrix}$

18.
$x(t) = 2e^{-t} + e^{5t}$
$y(t) = -2e^{-t} + 2e^{5t}$

7.6. Homogeneous Linear Systems with Constant Coefficients; Complex Eigenvalues

19. $\mathbf{x}(t) = c_1\begin{pmatrix}\cos(2t)\\ \cos(2t) + \sin(2t)\end{pmatrix}e^{-t} + c_2\begin{pmatrix}\sin(2t)\\ \sin(2t) - \cos(2t)\end{pmatrix}e^{-t}$

20. $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = -2\begin{pmatrix}-2\cos(3t)\\ \cos(3t) + 3\sin(3t)\end{pmatrix} - \begin{pmatrix}-2\sin(2t)\\ \sin(3t) - 3\cos(3t)\end{pmatrix} = \begin{pmatrix}4\cos(3t) + 2\sin(3t)\\ -7\sin(3t) + \cos(3t)\end{pmatrix}$

21. The first eigenvalue/eigenvector pair gives the solution $\mathbf{x_1}(t) = \begin{pmatrix}1\\ 1\\ 1\end{pmatrix}e^{-2t}$.

The second eigenvalue/eigenvector pair gives the two solutions:

$$\mathbf{x_2}(t) = \begin{pmatrix}\cos(t) + \sin(t)\\ 2\cos(t)\\ \cos(t)\end{pmatrix}e^t, \quad \mathbf{x_3}(t) = \begin{pmatrix}-\cos(t) + \sin(t)\\ 2\sin(t)\\ \sin(t)\end{pmatrix}e^t$$

7.8. Homogeneous Linear Systems with Constant Coefficients; Repeated Eigenvalues

22. $\mathbf{x}(t) = c_1\begin{pmatrix}1\\ 3\end{pmatrix}e^{-2t} + c_2\left(\begin{pmatrix}-1\\ 0\end{pmatrix} + \begin{pmatrix}3\\ 1\end{pmatrix}t\right)e^{-2t} = c_1\begin{pmatrix}1\\ 3\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}-1 + 3t\\ t\end{pmatrix}e^{-2t}$

23. $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = 2\begin{pmatrix}1\\ -1\end{pmatrix}e^t - 3\begin{pmatrix}1/3 + t\\ -t\end{pmatrix}e^t = \begin{pmatrix}1 - 3t\\ -2 + 3t\end{pmatrix}e^t$