MAT265 Review Problems for Exam 2

Product and Quotient Rules

1. Suppose $f(x) = g(x)(3x^2 + 4\cos x + 2^4)$ with $g(0) = 5$ and $g'(0) = 2$. Find $f'(0)$.
2. Suppose the derivative of f exists. Assume that $f(2) = 2$ and $f'(2) = 3$. Let $g(x) = x^2 f(x)$, $h(x) = \frac{f(x)}{x-3}$.
   a. Find an equation of the tangent line to $y = g(x)$ at $x = 2$.
   b. Find an equation of the tangent line to $y = h(x)$ at $x = 2$.
3. Suppose tangent to f at $x = 2$ is $y = 4x + 1$ and tangent to g at $x = 2$ is $y = 3x - 2$. Find the line tangent to the following curves at $x = 2$.
   a. $y = f(x)g(x)$
   b. $y = \frac{f(x)}{g(x)}$

Chain Rule using a table

4. Let $h(x) = f(g(x))$, $p(x) = g(f(x))$, $q(x) = g(f(e^{2x} + 1))$ and $l(x) = (2x - 3\sin x - 5g(x))^4$. Using the table to compute the following derivatives.
   a. $h'(3)$ b. $h'(2)$ c. $p'(4)$ d. $p'(2)$ e. $q'(0)$ f. $l'(0)$

x	0	1	2	3	4	5
$f(x)$	-1	0	3	5	1	0
$f'(x)$	1	5	3	-5	-8	-10
$g(x)$	2	4	5	1	3	2
$g'(x)$	-2	2	10	20	15	20

Derivative of composite functions

5. Suppose f is differentiable on [-2, 2] with $f'(0) = 3, f(1) = 5,$ and $f'(1) = 5$. Let $g(x) = f(\sin x)$. Evaluate the followings.
   a. $g'(0)$ b. $g'\left(\frac{\pi}{2}\right)$ c. $g'(\pi)$

Other Chain Rule

6. Solve.
   a. Suppose $[g(x)]^2 + 12x = x^2 g(x)$ and $g(4) = 12$. Find $g'(4)$.
   b. If $y = 2x^3 + 5x$ and $\frac{dx}{dt} = 3$, find $\frac{dy}{dt}$ when $x = 1$.

Horizontal tangent

7. Find all points on $[0, 2\pi]$ at which $f(x) = 2\sin x + \sin^2 x$ has horizontal tangent.

Tangent line using Implicit Differentiation

8. Find an equation of the tangent line to the curve at the given point.
   a. $x^2 + xy + y^2 = 7$; (2,1)
   b. $\cos(x - y) + \sin y = \sqrt{2}; \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$
   c. $(x^2 + y^2)^2 = \frac{25}{4} xy^2$; (1,2)

Second Derivatives

9. Find $\frac{d^2y}{dx^2}$.
   a. $\sin x + x^2 y = 1$
   b. $e^{2y} + x = y$

Related Rates

10. Sand falls from an overhead bin at a rate of 3 m^3/min and accumulates in a cone shaped pile, whose height is equal to twice the radius. How fast is the height of the cone rising when the height is 2 meters?
11. Two small planes approach an airport, one flying due west at 120 mi/hr and the other flying due north at 150 mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 mi from the airport and the northbound plane is 225 mi from the airport.
12. If a snowball melts so that its surface area decreases at the rate of $1\ cm^2/min$, find the rate at which the diameter decreases when the diameter is 10 cm. (Hint : Surface area $= 4\pi r^2$)
13. A 13-ft ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away from the wall at a rate of $0.5\ ft/s$. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 ft from the wall?

Linear Approximation

14. Use the linear approximation of the function $f(x) = \sqrt{9-x}$ at $a = 0$ to approximate the number $\sqrt{8.99}$.
15. Use Differentials (or linear approximation) to approximate the followings.
    a. $\ln(1.05)$ b. $(8.06)^{\frac{2}{3}}$

Calculating Error Using Differentials

16. The circumference of a sphere was measured to be 84 cm with a possible error in measurement of 0.2 cm.
    a. Use differentials to estimate the maximum error in the calculated surface area.
    b. Calculate the relative error.
    c. Calculate the percentage error.

Limits of Exponentials

17. Find the following limits:
    a. $\lim_{x\to\infty} \frac{2+7^{x+1}}{2-7^{x-1}}$ b. $\lim_{x\to\infty} x^2 e^{-x^2}$

18. Let $f(x) = e^{5/(x-4)}$. Find the following limits:
    a. $\lim_{x\to 4^-} f(x)$
    b. $\lim_{x\to 4^+} f(x)$

Inverse Functions

19. Consider $f(x) = \sqrt[3]{3-5x}$ and let $a = 2$.
    a. What is $f^{-1}(a)$?
    b. Compute $(f^{-1})'(a)$ using the formula $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.

20. Consider $f(x) = x^2 + \sin(x)$ for $-0.4 \le x \le 0.4$.
    a. Find $f^{-1}(0)$.
    b. Find $f'(x)$.
    c. Find $(f^{-1})'(0)$.

Derivatives of Natural Logarithms and Exponentials

21. Use logarithmic differentiation to find y' if:
    a. $y = \ln\left(\frac{e^{2x}\sqrt[3]{x-4x^5}}{(x^3-6x)^7}\right)$
    b. $e^{x+y} = xy$
    c. $f(x)^y = y^{f(x)}$
22. Suppose $f(x) = 1 + x^3 + e^{x-1}$. Find $(f^{-1})'(3)$. (Hint: Look at inverse function theorem in 3.2)
23. Suppose $f(x) = \cos(e^{10x})$. What is $f'(x)$? What is $f''(x)$?

Inverse Trigonometric Functions

24. Prove the followings.
    a. if $y = \cos^{-1}(x)$ then $y' = -\frac{1}{\sqrt{1-x^2}}$.
    b. if $y = \tan^{-1}(x)$ then $y' = \frac{1}{x^2+1}$.
    c. if $y = \cot^{-1}(x)$ then $y' = -\frac{1}{x^2+1}$.
25. Find $\lim_{x\to\infty} \arctan\left(\frac{3x^2+\sin(x)}{x+1}\right)$.

l'Hospital's Rule

26. Find the following limits using l'Hospital's Rule:
    a. $\lim_{\theta\to\frac{\pi}{2}}\left(\frac{\cos(5\theta-2k\pi)-2\theta+\pi}{14\theta-7\pi}\right)$ where k is an integer.
    b. $\lim_{x\to 0+} x^{-2}\sin(x)$
    c. $\lim_{x\to 0}\frac{2e^x-3x^2-2}{3e^x+5x-3}$.
    d. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.
    e. $\lim_{x\to\infty} x^{\frac{\ln 3}{5+\ln x}}$.

Note: There is a reasonable assumption that most of these answers are not incorrect.

1. 40
2. a) $y = 20x - 32$ b) $y = -5x + 8$
3. a) $y = 43x - 50$ b) $y = -\frac{11}{16}x + \frac{29}{8}$
4. a) 100 b) -100 c) -16 d) 60 e) 120 f) -36000
5. a. 3 b. 0 c. -3
6. a.21/2 b.33
7. $\left(\frac{\pi}{2}, 3\right), (\frac{3\pi}{2}, -1)$
8. a) $y = -\frac{5}{4}x + \frac{7}{2}$ b) $y = \frac{1}{2}x$ c) $y = \frac{1}{3}x + \frac{5}{3}$
9. a) $\frac{x\sin x + 4\cos x + 6xy}{x^3}$ b) $\frac{4e^{2y}}{(1-2e^{2y})^3}$
10. $\frac{3}{\pi}$ meters per minute
11. $-\frac{1230}{\sqrt{41}} \approx -192$ mi/hr : distance between two planes is decreasing at a rate of 192mi/hr
12. $\frac{-1}{20\pi}$ cm/min or decreasing at a rate of $\frac{1}{20\pi}$ cm/min
13. $-\frac{5}{24}$ ft/sec : the top of the ladder slides down the wall at 5/24 feet per second
14. $\frac{1799}{600} \approx 2.9983$
15. a) 0.05 b) 4.02
16. a) $\frac{168}{5\pi} \approx 10.7\ cm^2$ b) $\frac{1}{210} \approx 0.0048$ c) 0.48%
17. a) -49 b) 0
18. a) 0 b) ∞
19 . a) -1 b) $\frac{-12}{5}$
20. a) 0 b) 2x+cos(x) c) 1
21. a) $2 + \frac{1-20x^4}{3(x-4x^5)} - \frac{21x^2-42}{x^3-6x}$ b) $\frac{1-x^{-1}}{y^{-1}-1}$ c) $f'(x)\left(\frac{y/f(x)-\ln(y)}{f(x)/y-\ln(f(x))}\right)$
22. ¼
23. $-10e^{10x}\sin(e^{10x})$, $-100e^{10x}(e^{10x}\cos(e^{10x}) + \sin(e^{10x}))$
24.
25. $\frac{\pi}{2}$
26. a) -1/2 b) ∞ c) 1/4 d) e e) 3