Graph Theory Qualifier

May 10, 2011

1. (15 pts.) Prove that for all graphs $G$ and positive integers $a$ and $b$ if $|G| \geq 2^{a+b-2}$ then $\omega(G) \geq a$ or $\alpha(G) \geq b$.

2. (15 pts.) Recall that $T_{n,r}$ is the $r$-partite graph on $n$ vertices such that every part has size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$. Prove that among all graphs $G = (V,E)$ on $n$ vertices with $\omega(G) \leq r$, the one with the most edges is $T_{n,r}$. (You may use the fact that this is true for all $r$-partite graphs.)

3. (15 pts.) Let $G$ be an $X,Y$-bigraph with $|X| = |Y| = k$ and $\delta(G) \geq \frac{k}{2}$. Prove that $G$ has a 1-factor.

4. (25 pts.) Let $G$ be a planar bipartite graph.
   a) Use Euler’s Formula to prove that $|G| \leq 2|G| - 4$.
   b) Suppose $G$ has an orientation with $d^+(r) \geq 3$ for some vertex $r \in V$. Let $W \subseteq V$ be the set of all vertices $w$ (including $r$) such that there is a directed path from $r$ to $w$. Prove that $H = G[W]$ has a vertex $v$ with $d_H^+(v) \leq 1$.
   c) Show that $G$ has an orientation with maximum out-degree at most 2. [Hint: Argue by induction and consider reversing a directed path.]
   d) It is known that every directed graph without a directed odd cycle has a kernel. Use this to show that $G$ is 3-list colorable.
   e) Construct a small example (including lists) of a graph that is bipartite and planar, but not 2-list colorable.

5. (15 pts.) Let $G = (V,E)$ be a nonplanar 3-connected graph with at least six vertices. Show that $G$ contains a subdivision of $K_{3,3}$.

6. (15 pts.) Let $G = (V,E)$ be a 2-edge-connected graph. Show that $G$ has a spanning 2-edge-connected subgraph with $|V| \leq 2|G| - 2$. [Hint: Start with a spanning tree $T$ and add a small set $F \subseteq E$ to $T$ so that $T + F - e$ is connected for every $e \in E(T)$.]