Exam 3 Key

1) **C** Note: $(-1,1)$ is the largest open interval of convergence, the series converges for all $x \in (-1,1]$.
2) **B**
3) **B**
4) **B**
5) **B**
6) **A**
7) **C** Note: $\left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(1 - 8x^2 + \frac{32}{3}x^4 - \cdots\right) = 1 - 9x^2 + \frac{115}{6}x^4 - \cdots$
8) **D** Note: we only get part of the Cartesian curve, specifically, $y = \frac{1}{x}$, $0 < x \leq 1$.
9) **C**
10) **B**
11) **E**
12) **B** Note: simplifies to $\int_0^1 \sqrt{36t^2 + 36t^4}\, dt = \int_0^1 6t\sqrt{1+t^2}\, dt$
13) Using series,

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - x}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + \cdots\right) = \frac{1}{6} + 0 + 0 + \cdots = \frac{1}{6}$$

14) Using the known series for $\sin(x)$,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

So

$$x \cdot \sin(x^2) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)(x^{4n+2})}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!}$$

Or

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$x \cdot \sin(x^2) = x\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots + \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= x^3 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^n x^{4n+3}}{(2n+1)!} + \cdots$$

15) Ratio Test

$$a_m = \frac{m^5}{3^m} \Rightarrow a_{m+1} = \frac{(m+1)^5}{3^{m+1}}$$

So

$$L = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{(m+1)^5}{3^{m+1}} \cdot \frac{3^m}{m^5}\right| = \lim_{m\to\infty}\left(\frac{m+1}{m}\right)^5 \cdot \frac{1}{3} = (1)^5 \cdot \frac{1}{3} = \frac{1}{3}$$

$L < 1$ implies the series converges (absolutely).

16) Change the problem to: Find the first four nonzero terms of the Maclaurin series for $f(x) = \sqrt[3]{8+x}$. Use your result to approximate $\sqrt[3]{8.05}$.

$$f(x) = (8+x)^{1/3} \quad \Rightarrow \quad f(0) = (8)^{1/3} = 2 \quad \Rightarrow \quad c_0 = \frac{f(0)}{0!} = \frac{2}{1} = 2$$

$$f'(x) = \frac{1}{3}(8+x)^{-\frac{2}{3}} \quad \Rightarrow \quad f'(0) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12} \quad \Rightarrow \quad c_1 = \frac{f'(0)}{1!} = \frac{1/12}{1} = \frac{1}{12}$$

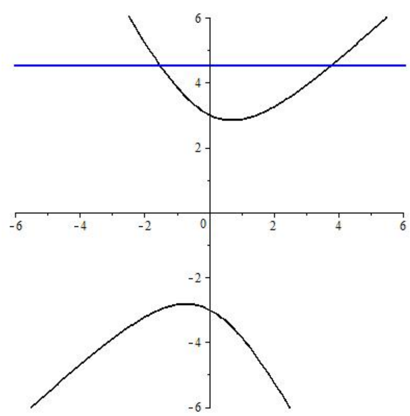
$$f''(x) = -\frac{2}{9}(8+x)^{-\frac{5}{3}} \quad \Rightarrow \quad f''(0) = -\frac{2}{9}(8)^{-\frac{5}{3}} = -\frac{1}{144} \quad \Rightarrow \quad c_2 = \frac{f''(0)}{2!} = \frac{-1/144}{2} = -\frac{1}{288}$$

$$f'''(x) = \frac{10}{27}(8+x)^{-\frac{8}{3}} \quad \Rightarrow \quad f'''(0) = \frac{10}{27}(8)^{-\frac{8}{3}} = \frac{5}{3456} \quad \Rightarrow \quad c_3 = \frac{f'''(0)}{3!} = \frac{5/3456}{6} = \frac{5}{20736}$$

So $f(x) = \sqrt[3]{8+x} \approx 2 + \frac{1}{12}x - \frac{1}{288}x^2 + \frac{5}{20736}x^3$ and

$f(0.05) = \sqrt[3]{8.05} \approx 2 + \frac{1}{12}(0.05) - \frac{1}{288}(0.05)^2 + \frac{5}{20736}(0.05)^3 \approx 2.004158$

17) Graph of $x_1 = t - \frac{1}{t}, y_1 = t + \frac{2}{t}$ and $x_2 = t - \frac{1}{t}, y_2 = 4.5$ is shown below.



Area is $\int_a^b (y_{top} - y_{bottom})dx$; to put in terms of the parameter t, we need the t-values of the intersection of the two curves:

$$t + \frac{2}{t} = \frac{9}{2} \Rightarrow 2t^2 + 4 = 9t \Rightarrow 2t^2 - 9t + 4 = 0$$

$$\Rightarrow (2t-1)(t-4) = 0 \Rightarrow t = \frac{1}{2} \text{ and } t = 4$$

Also, $dx = \left(1 + \frac{1}{t^2}\right)dt$, so the area is

$$\int_{1/2}^{4} \left(\frac{9}{2} - \left(t + \frac{2}{t}\right)\right)\left(1 + \frac{1}{t^2}\right)dt \cong 5.574$$

(Note, the exact value is $\frac{189}{16} - 9\ln(2)$. Also, the parameterization of the horizontal line is given so that the two curves have the same dx.)

18) Using $x = r\cos(\theta)$, $y = r\sin(\theta)$, with $r = 2\theta$ as the given curve, we have $x = 2\theta\cos(\theta)$, $y = 2\theta\sin(\theta)$. So

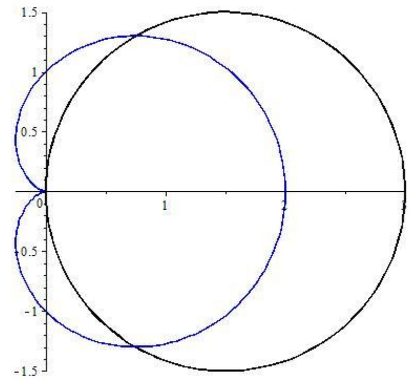
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin(\theta) + 2\theta\cos(\theta)}{2\cos(\theta) + 2\theta(-\sin(\theta))}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{0 + 2\pi(-1)}{2(-1) + 0} = \pi$$

19) The two curves are graphed below:

The two curves intersect when $3\cos(\theta) = 1 + \cos(\theta)$

$\Rightarrow 2\cos(\theta) = 1$ so $\cos(\theta) = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}$



Area between the curves is $A = \int_{\theta_1}^{\theta_2} \frac{1}{2}\left(r_{outer}^2 - r_{inner}^2\right)dr$

$$= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(9\cos^2(\theta) - (1 + 2\cos(\theta) + \cos^2(\theta))\right)d\theta$$

$\cos(\theta)$ is even, so we can double and integrate from 0 to $\frac{\pi}{3}$:

$$= \int_0^{\pi/3} (8\cos^2(\theta) - 2\cos(\theta) - 1)d\theta$$

$$= \int_0^{\pi/3} \left(8\frac{1+\cos(2\theta)}{2} - 2\cos(\theta) - 1\right)d\theta = \int_0^{\pi/3} (3 + 4\cos(2\theta) - 2\cos(\theta))d\theta$$

$$= [3\theta + 2\sin(2\theta) - 2\sin(\theta)]_0^{\pi/3} = \left[\pi + 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\right] - [0] = \pi$$

20) We have

$\frac{dy}{dt} = 3e^{-t} - 3te^{-t}$ and $\frac{dx}{dt} = 15e^t + 15te^t$. We have a horizontal tangent where $\frac{dy}{dt} = 0$. That is,

$3e^{-t} - 3te^{-t} = 0 \Rightarrow 3e^{-t}(1 - t) = 0 \Rightarrow t = 1$ (note: $dx/dt \neq 0$ at $t = 1$).

At $t = 1$, $x = 15e$, $y = 3e^{-1} = \frac{3}{e}$, so the highest point on the graph is $\left(15e, \frac{3}{e}\right)$.

(Refer to the graph to verify this point is indeed a maximum, or note for $t > 0$, (positive y-values for the graph), $\frac{dy}{dx} > 0$ for $0 < t < 1$, and $\frac{dy}{dx} < 0$ for $t > 1$.)