PRINT NAME: ________________________________

APM 503 / MAT 570 Final Exam and Qualifier Exam Friday, 12/14/12

Print email: ________________________________

☐ CHECK THIS BOX IF YOU WANT THE EXAM TO COUNT AS A QUALIFIER

Time limit: 1 hour and 50 minutes

Write your solutions on the blank pages provided. Write on only one side of each page, and leave reasonable margins (the papers will be photocopied before grading).

Start every problem on a new page. DO NOT STAPLE OR FOLD CORNERS OF THE PAGES.

Print your name clearly in the upper right corner of each page that you submit, including this cover page. Turn in your exam with this cover page on top. You may keep the second page (of the problems).

If you are enrolled in the course, the exam will be graded automatically as the final exam. Be sure to check the box on this cover page if you want the exam to be graded also as Part 1 of the Qualifier Exam in Real Analysis.

Write your final solutions clearly and in logical order, using complete sentences and correct English. Be sure to verify all hypotheses of any theorem that you use, and also give the name of the theorem if it has one.

You may cite results proved in lecture or in homework, unless these are part of what you are asked to prove. You may ignore any hints, but you must follow all explicit instructions. All problems have equal value, and in a problem with more than one part, all parts have equal value.

No notes, books, calculators, or other electronic devices are allowed.

TURN OFF CELLPHONES, AND PUT THEM AWAY FOR THE DURATION OF THE EXAM.

As a courtesy to the rest of the class, please do not leave, or pack up your things, during the last 15 minutes of the exam period.
1. Let \( f, f_1, f_2, \ldots : \mathbb{R} \to \mathbb{C} \), and let \( E \subseteq \mathbb{R} \).

(a) Complete the definition (without using the symbol \( \| \cdot \| \)):
\[ f_n \to f \text{ uniformly on } E \text{ if . . .} \]

(b) Complete the definition (without using the symbol \( \| \cdot \| \)):
\[ f_n \to f \text{ in } L^1(\mathbb{R}) \text{ if . . .} \]

(c) Give an example (with proof) of a sequence in \( L^1(\mathbb{R}) \) that converges uniformly on \( \mathbb{R} \), but does not converge in \( L^1(\mathbb{R}) \).

(d) Give an example (with proof) of a sequence in \( L^1(\mathbb{R}) \) that converges in \( L^1(\mathbb{R}) \), and converges uniformly on each compact interval in \( \mathbb{R} \), but does not converge uniformly on \( \mathbb{R} \).

2. Let \( f \in L^1(\mathbb{R}^n) \) and let \( g \in C_0(\mathbb{R}^n) \). Prove that \( f \ast g \) vanishes at infinity. (In fact, \( f \ast g \in C_0(\mathbb{R}^n) \), but you do not have to prove continuity for this problem).

(Hint: consider an appropriate subspace, of both \( L^1(\mathbb{R}^n) \) and of \( C_0(\mathbb{R}^n) \), that is dense in both.)

3. Investigate (with proof) \( \lim_{n \to \infty} \int_0^\infty ne^{-x^2} \sin(\pi e^{x/n}) \, dx \).

(Hints: 1) Apply L’Hôpital’s rule to \( n \sin(\pi e^{x/n}) \) as a function of \( n \).
2) Apply the mean value theorem to \( \sin(\pi e^{x/n}) \) as a function of \( x \).)

4. For \( f \in L^1(\mathbb{R}) \), let \( f^{*k} = f \ast f \ast \cdots \ast f \). Let \( \phi(x) = e^{-\pi x^2} \). Prove that there exists \( f \in L^1(\mathbb{R}) \) such that \( f^{*k} = \phi \). (You may use the fact that the Fourier transform of \( e^{-ax^2} \) is \( \sqrt{\pi/a} e^{-\pi^2 x^2} \).

5. Define \( \Phi : L^2(\mathbb{R}) \to L^1(\mathbb{R}) \) by \( \Phi(f)(x) = f(x)^2 - f(x-1)f(x+1) \). Prove that \( \Phi \) is continuously differentiable on \( L^2(\mathbb{R}) \).