MAT 271

Exam 3

SAMPLE

NOTE: This is only a sample to give an idea of form and length it is not meant to indicate exact content of your exam. The actual material covered on each exam varies from semester to semester.

1. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If $\displaystyle\sum_{n=0}^{\infty} |a_n|$ converges, then the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

   True **False**

   (b) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/5}}$ diverges.

   True **False**

   (c) Consider the series $\displaystyle\sum_{n=0}^{\infty} a_n$. If $\displaystyle\lim_{n\to\infty} a_n = 0$, then the series is convergent.

   True **False**

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

   True **False**

   (e) Consider the series $\displaystyle\sum_{n=0}^{\infty} ar^n, a \neq 0$. If $|r| < 1$, then the series diverges.

   True **False**

2. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

   a) Use the ratio test to determine whether the series converges or diverges:

   $$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}.$$

   NOTE: Your work will be checked for use of the ratio test so label this work on your scratch paper.

   b) Find the radius of convergence of the series

   $$\sum_{n=0}^{\infty} \cos(n\pi)\left(\frac{x}{3}\right)^n.$$

   c) Using a known power series, find the power series representation of the function in Σ-notation.

   $$f(x) = x^3 e^{2x}.$$

   d) Find the interval of convergence of the series

   $$\sum_{n=1}^{\infty} \frac{x^{3n}}{n^{1/3}}.$$

   e) Use the root test to determine whether the series converges or diverges:

   $$\sum_{n=1}^{\infty} \frac{n^2}{e^n}.$$

   NOTE: Your work will be checked for use of the root test so label this work on your scratch paper.

3. (10 points) Find the Taylor series of $f(x) = \sin x$ at $a = \frac{\pi}{3}$. Remember to show at least four non-zero terms.

4. (10 points) Use the known power series for $\frac{1}{1-x}$ to build a power series representation for $f(x) = \frac{x^2}{(1-2x)^2}$. Hint: Consider integration or differentiation.

Solutions

1. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If $\sum_{n=0}^{\infty} |a_n|$ converges, then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

   [✓ **True**] **False**

   (b) The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/5}}$ diverges.

   [✓ **True**] **False**

   (c) Consider the series $\sum_{n=0}^{\infty} a_n$. If $\lim_{n\to\infty} a_n = 0$, then the series is convergent.

   True [✓ **False**]

   (d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

   [✓ **True**] **False**

   (e) Consider the series $\sum_{n=0}^{\infty} ar^n, a \neq 0$. If $|r| < 1$, then the series diverges.

   True [✓ **False**]

2. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

a) Use the ratio test to determine whether the series converges or diverges:

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}.$$

NOTE: Your work will be checked for use of the ratio test so label this work on your scratch paper.

$$\boxed{\text{converges}}$$

b) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \cos(n\pi) \left(\frac{x}{3}\right)^n.$$

$$\boxed{3}$$

c) Using a known power series, find the power series representation of the function in Σ-notation.

$$f(x) = x^3 e^{2x}.$$

$$\boxed{\sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+3}}$$

d) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{x^{3n}}{n^{1/3}}.$$

$$\boxed{[-1, 1)}$$

e) Use the root test to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}.$$

NOTE: Your work will be checked for use of the root test so label this work on your scratch paper.

$$\boxed{\text{converges}}$$

3. (10 points) Find the Taylor series of $f(x) = \sin x$ at $a = \frac{\pi}{3}$. Remember to show at least four non-zero terms.

Solution:

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$	$\frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!}$
0	$f(x) = \sin x$	$\frac{\sqrt{3}}{2}$	$\frac{\frac{\sqrt{3}}{2}}{0!} = \frac{\sqrt{3}}{2}$
1	$f'(x) = \cos x$	$\frac{1}{2}$	$\frac{\frac{1}{2}}{1!} = \frac{1}{2}$
2	$f''(x) = -\sin x$	$-\frac{\sqrt{3}}{2}$	$\frac{-\frac{\sqrt{3}}{2}}{2!} = -\frac{\sqrt{3}}{2 \cdot 2!}$
3	$f”'(x) = -\cos x$	$-\frac{1}{2}$	$\frac{-\frac{1}{2}}{3!} = -\frac{1}{2 \cdot 3!}$

The Taylor seires for $\sin x$ at $\frac{\pi}{3}$ is:

$$\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

4. (10 points) Use the known power series for $\dfrac{1}{1-x}$ to build a power series representation for $f(x) = \dfrac{x^2}{(1-2x)^2}$. Hint: Consider integration or differentiation.

Solution: Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \Longrightarrow \quad \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

and

$$\frac{1}{2}\frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{1}{2}\frac{2}{(1-2x)^2} = \frac{1}{(1-2x)^2}$$

We get

$$\begin{aligned}\frac{x^2}{(1-2x)^2} &= x^2\frac{1}{(1-2x)^2} = \frac{x^2}{2}\cdot\frac{d}{dx}\left(\frac{1}{1-2x}\right)\\ &= \frac{x^2}{2}\frac{d}{dx}\sum_{n=0}^{\infty} 2^n x^n = \frac{x^2}{2}\sum_{n=1}^{\infty} 2^n \frac{d}{dx} x^n\\ &= \frac{x^2}{2}\sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=1}^{\infty} \frac{x^2}{2}\cdot 2^n n x^{n-1}\\ &= \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1}\end{aligned}$$

Scratch area

2. Work.

a)

$$\lim_{n\to\infty} \frac{(n+1)!\,(n+1!}{(2n+2)!} \cdot \frac{(2n)!}{n!\,n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)n!(n+1)n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{n!\,n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

b)

$$\sum_{n=0}^{\infty} \cos(n\pi)\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{3}\right)^n$$

This is geometric with

$$|r| = \left|\frac{(-1)^n x}{3}\right| = \frac{|x|}{3},$$

so it converges for

$$\frac{|x|}{3} < 1 \quad \Longrightarrow \quad |x| < 3 \Longrightarrow R = 3$$

c)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Longrightarrow \quad e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

$$x^3 e^{2x} = x^3 \cdot e^{2x} = x^3 \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} x^3$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+3}$$

d)

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{x^n}{n^{1/3}}\right|}=\lim_{n\to\infty}\frac{\sqrt[n]{|x|^n}}{\sqrt[n]{n^{1/3}}}=\lim_{n\to\infty}\frac{|x|}{(\sqrt[n]{n})^{1/3}}=\frac{|x|}{1^{1/3}}=\frac{|x|}{1}=|x|$$

It converges when

$$|x|<1 \quad\Longrightarrow\quad -1<x<1.$$

If $x=-1$, the series is

$$\sum_{n=1}^{\infty}\frac{(-1)^{3n}}{n^{1/3}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$$

Since $\lim\limits_{n\to\infty}\frac{1}{n^{1/3}}=0$, the series is alternating, and $\frac{1}{(n+1)^{1/3}}<\frac{1}{n^{1/3}}$ the series converges by AST.

If $x=1$, the series is

$$\sum_{n=1}^{\infty}\frac{(1)^{3n}}{n^{1/3}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$$

Since this is a p-series with $p<1$, so it diverges. Hence the IOC is $[-1,1)$.

e)

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{e^n}}=\lim_{n\to\infty}\frac{\sqrt[n]{n^2}}{\sqrt[n]{e^n}}=\lim_{n\to\infty}\frac{(\sqrt[n]{n})^2}{e}=\frac{1}{e}<1$$