MAT 271

Exam 2

SAMPLE

NOTE: This is only a sample to give an idea of form and length it is not meant to indicate exact content of your exam. The actual material covered on each exam varies from semester to semester.

1. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) The area of the region bounded by the graphs of $f(x) = x^3$ and $g(x) = 4x$ is $2\int_0^2 (4x - x^3)dx$.

   True **False**

   (b) The sequence $\left\{\frac{6(n+2)!}{(n+4)!}\right\}_{n=1}^{\infty}$ converges.

   True **False**

   (c) If an object is moving along a straight line with velocity v, then the total distance that the object has travelled over the interval $[2, 6]$ is $\int_2^6 v(t)\,dt$.

   True **False**

   (d) If the sequence $\{a_n\}_{n=1}^{\infty}$ is defined by the formula $a_n = \frac{(-1)^n + 2}{3^n}$, then the second term of the sequence is $\frac{1}{3}$.

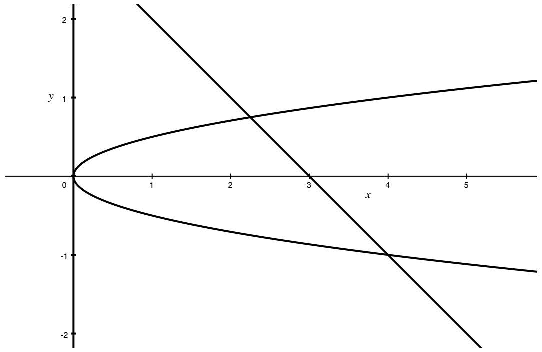
   True **False**

   (e) If $a_n > 5$ and $a_{n+1} \leq a_n$ for all n, then $\{a_n\}_{n=1}^{\infty}$ diverges.

   True **False**

2. (20 points) The parts of this problem are independent. In this problem there is no partial credit, and you do not need to show your work.

   a) Set up, DO NOT EVALUATE, a single integral for the area bounded by the curves $x = 4y^2$ and $y = 3 - x$.



   b) Write and simplify, but DO NOT EVALUATE, an integral to find the length of the curve given by,

$$y = e^{-3x}, \quad 3 \leq x \leq 7.$$

   c) Determine whether the sequence

$$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \frac{32}{27}, \ldots \right\}$$

   converges or diverges. If it converges, find the limit.

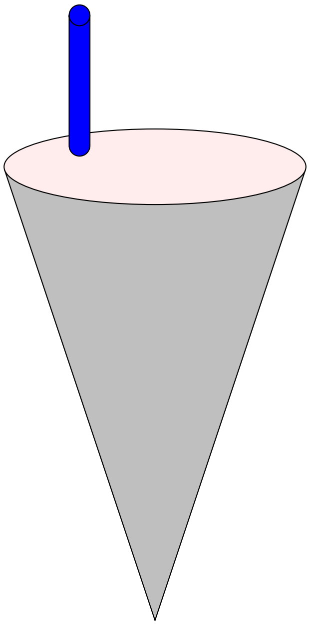
   d) The velocity function (in meters per second) for a particle moving along a line is given by

$$v(t) = 8 - 2t, \quad 0 \leq t \leq 6.$$

   Find the total distance traveled (in meters) by the particle.

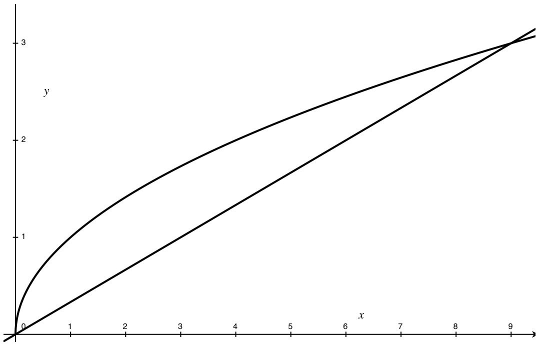
   e) Determine whether the sequence $\left\{ \dfrac{n^2}{3 + 5n^2} \right\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

3. (10 points) A paper cup in the shape of an inverted cone of radius 1/6 ft is filled with a strawberry milkshake to the top of the cup; the height of the cup is 1/2 ft. SET UP, do not evaluate, an integral for the work required to suck all the milkshake out through a straw that extends 1/3 ft over the top. You *must* explain your set-up. Assume that milkshake weighs 72 lb/ft^3. DO NOT EVALUATE THE INTEGRAL. In the illustration given show your axes and label all information; *without this we will not grade the problem.*

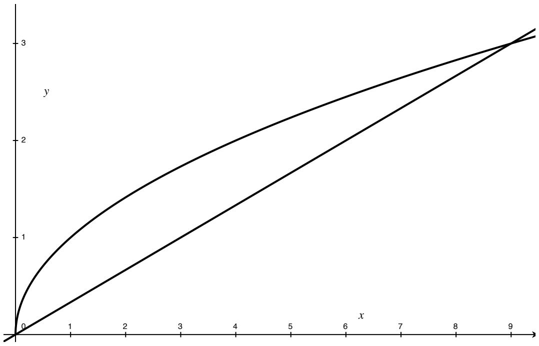


4. (10 points) The region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$ is rotated about the x-axis.

   a) Use the washer method to set up, DO NOT EVALUATE, a single integral to find the volume of the resulting solid. Illustrate the method on the graph provided.



   b) Use the shell method to set up, DO NOT EVALUATE, a single integral to find the volume of the resulting solid. Illustrate the method on the graph provided.



Solutions

1. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) The area of the region bounded by the graphs of $f(x) = x^3$ and $g(x) = 4x$ is $2\int_0^2 (4x - x^3)dx$.

   [✓ **True**] **False**

   (b) The sequence $\left\{\dfrac{6(n+2)!}{(n+4)!}\right\}_{n=1}^{\infty}$ converges.

   [✓ **True**] **False**

   (c) If an object is moving along a straight line with velocity v, then the total distance that the object has travelled over the interval $[2, 6]$ is $\int_2^6 v(t)\,dt$.

   True [✓ **False**]

   (d) If the sequence $\{a_n\}_{n=1}^{\infty}$ is defined by the formula $a_n = \dfrac{(-1)^n + 2}{3^n}$, then the second term of the sequence is $\dfrac{1}{3}$.

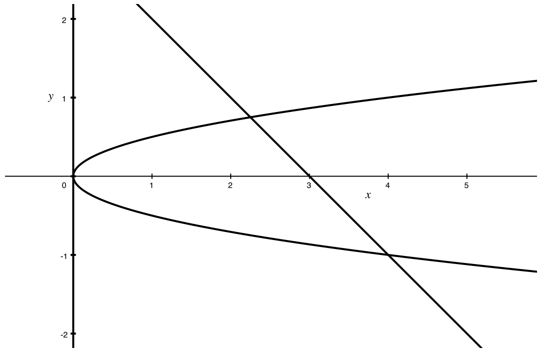
   [✓ **True**] **False**

   (e) If $a_n > 5$ and $a_{n+1} \leq a_n$ for all n, then $\{a_n\}_{n=1}^{\infty}$ diverges.

   True [✓ **False**]

2. (20 points) The parts of this problem are independent. In this problem there is no partial credit, and you do not need to show your work.

   a) Set up, DO NOT EVALUATE, a single integral for the area bounded by the curves $x = 4y^2$ and $y = 3 - x$.



$$\boxed{\int_{-1}^{3/4} (3 - y - 4y^2)dy}$$

   b) Write and simplify, but DO NOT EVALUATE, an integral to find the length of the curve given by,

$$y = e^{-3x}, \quad 3 \leq x \leq 7.$$

$$\boxed{\int_{3}^{7} \sqrt{1 + 9e^{-6x}}\, dx}$$

   c) Determine whether the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{8}{9}, \frac{32}{27}, \ldots\right\}$$

   converges or diverges. If it converges, find the limit.

$$\boxed{\text{diverges}}$$

   d) The velocity function (in meters per second) for a particle moving along a line is given by

$$v(t) = 8 - 2t, \quad 0 \leq t \leq 6.$$

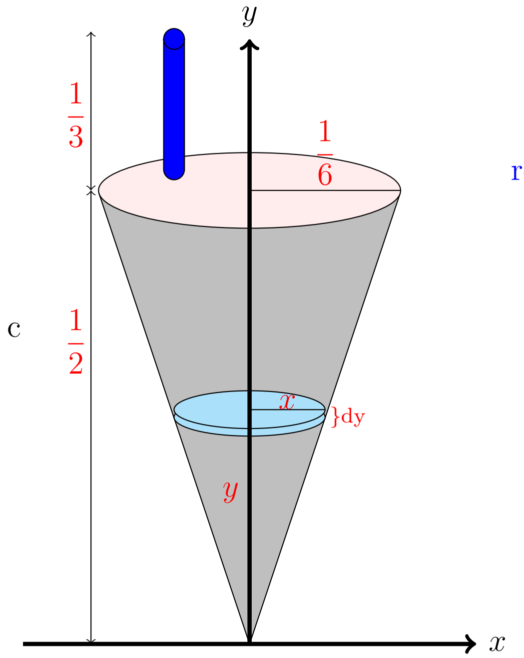
   Find the total distance traveled (in meters) by the particle.

$$\boxed{20 \text{ m}}$$

   e) Determine whether the sequence $\left\{\frac{n^2}{3 + 5n^2}\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

$$\boxed{\text{converges to } \frac{1}{5}}$$

3. (10 points) A paper cup in the shape of an inverted cone of radius 1/6 ft is filled with a strawberry milkshake to the top of the cup; the height of the cup is 1/2 ft. SET UP, do not evaluate, an integral for the work required to suck all the milkshake out through a straw that extends 1/3 ft over the top. You *must* explain your set-up. Assume that milkshake weighs 72 lb/ft^3. DO NOT EVALUATE THE INTEGRAL. In the illustration given show your axes and label all information; *without this we will not grade the problem.*



relationship between x and y

$$\frac{x}{y} = \frac{1/6}{1/2}$$
$$\frac{x}{y} = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{3}$$
$$x = \frac{1}{3}y$$

Area of slice: $\pi r^2 = \pi x^2$ ft^2

Volume of slice: $\pi x^2 dy$ ft^3, now we need this in terms of y, from the relationship found above: $\pi\left(\frac{1}{3}y\right)^2 dy = \frac{\pi}{9}y^2dy$ ft^3

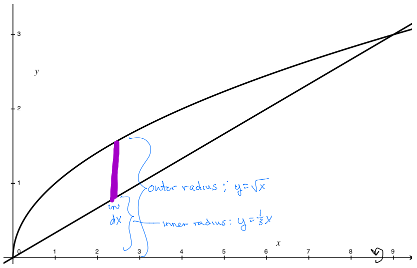
Force = Weight of slice = weight density × volume: $72\frac{\text{lb}}{\text{ft}^3}\cdot\frac{\pi}{9}y^2dy$ ft^3 $= 8\pi y^2dy$ lb

Work on slab = force × distance: $\left(\frac{5}{6}-y\right)8\pi y^2dy$ ft-lb Total work to move all slabs: any slab could have been between 0 and the top of the milkshake, 1/2 ft

$$W = 8\pi\int_0^{1/2}\left(\frac{5}{6}-y\right)8\pi y^2dy \text{ ft-lb}$$

4. (10 points) The region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{3}x$ is rotated about the x-axis.

a) Use the washer method to set up, DO NOT EVALUATE, a single integral to find the volume of the resulting solid. Illustrate the method on the graph provided.



To get a washer we need the slice of area to be perpendicular to the axis of rotation.

thickness of slice: dx

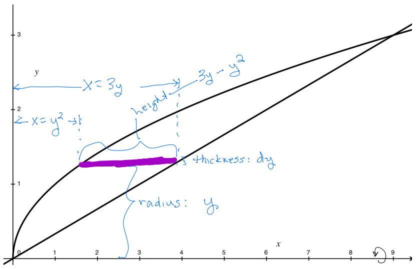
surface area of slice: $\pi\left((\sqrt{x})^2 - \left(\frac{x}{3}\right)^2\right)$

volume of slice: $\pi\left(x - \frac{x^2}{9}\right)dx$

intersection of curves: $\sqrt{x} = \frac{x}{3} \implies x = 0, 9$

$$\boxed{V = \pi\int_0^9 \left(x - \frac{x^2}{9}\right)dx}$$

b) Use the shell method to set up, DO NOT EVALUATE, a single integral to find the volume of the resulting solid. Illustrate the method on the graph provided.



For shells we need the slice of area to be parallel to the axis of rotation.

thickness of slice: dy

height slice: $3y - y^2$

radius of shell: y

volume of shell: $2\pi y(3y - y^2)dy$

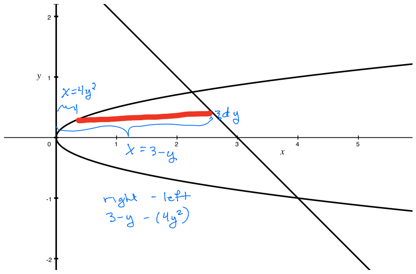
limits: $x = 0, 9 \implies y = 0, 3$

$$\boxed{V = 2\pi\int_0^3 \left(y(3y - y^2)\right)dy}$$

Scratch area

2. Work.

a)



$$3 - y = 4y^2 \implies (y+1)(4y-3) = 0$$

$$\implies y = -1, \frac{3}{4}$$

$$\int_{-1}^{3/4} (3 - y - 4y^2) dy$$

b) arc length is given by: $L = \int_a^b ds$; here

$$ds = \sqrt{1 + (y')^2} = \sqrt{1 + (-3e^{-3x})2}$$

so

$$\int_3^7 \sqrt{1 + 9e^{-6x}}\, dx$$

c) the sequence is geometric with $r = \frac{4}{3} > 1$, so it diverges

d) need to find zeros of v and see where $v > 0$ and $v < 0$

$$v = 0 \implies t = 4 \qquad \text{and if } t > 4, v < 0$$

$$\int_0^6 |v(t)|\, dt = \int_0^4 v(t)\, dt + \int_4^6 (-v(t))\, dt = \int_0^4 (8 - 2t)\, dt + \int_4^6 (-8 + 2t)\, dt$$
$$= (8t - t^2)\Big|_0^4 + (-8t + t^2)\Big|_4^6 = 20 \text{ m}$$

e)

$$\lim_{n\to\infty} \left(\frac{n^2}{3 + 5n^2} \right) = \frac{1}{5}$$

limit properties applied to sequences