MAT 270

Exam 3

SAMPLE

NOTE: This is only a sample to give an idea of form and length it is not meant to indicate exact content of your exam. The actual material covered on each exam varies from semester to semester.

1. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If the average rate of change of a differentiable function on $[0, 1]$ is zero, then its instantaneous rate of change must be zero somewhere between 0 and 1.

   TRUE FALSE

   (b) If $f'(c) = 0$ and $f''(c) = 12$, then c is a local minimum for f.

   TRUE FALSE

   (c) If

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

   then f grows faster than g. TRUE FALSE

   (d) If f is decreasing on the interval $[-1, 5]$, then the left Riemann Sum is an underestimate of the area under the curve between $x = -1$ and $x = 5$.

   TRUE FALSE

   (e) If f and g are increasing on an interval I, then $f + g$ is increasing on I.

   TRUE FALSE

2. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

(a) Evaluate the indefinite integral

$$\int \left(e^{2x} - \frac{3}{\sqrt{1-x^2}} - 5\sin(x) \right) dx.$$

(b) Use L'Hospital's Rule to find the limit:

$$\lim_{x \to 0} \left(\frac{\cos x - 1}{3x^2} \right)$$

NOTE: work will be checked for the use of L'Hospital's Rule so label this work on your scratch paper.

(c) Suppose

$$\int_7^2 f(x)dx = -7 \quad \text{and} \quad \int_2^{12} f(x)dx = 5.$$

Evaluate $\int_7^{12} f(x)dx$

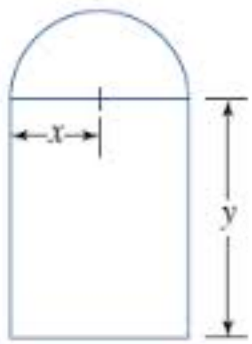
(d) Find the linear approximation to $f(x) = \sqrt[3]{x}$ at $a = 8$.

(e) Given that the acceleration of a particle is

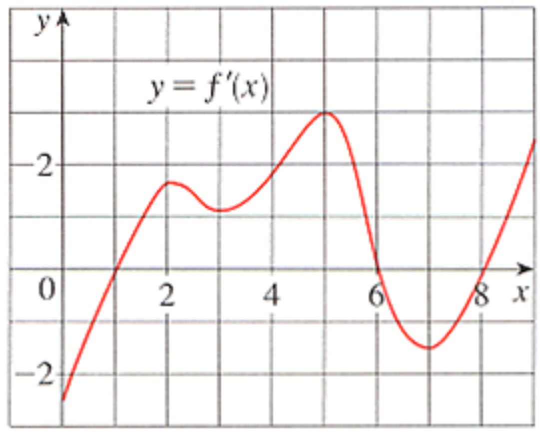
$$a(t) = 3t + 5 \text{ m/sec}^2,$$

find the velocity of the particle if the velocity at 2 seconds is 5 m/sec.

3. (10 points) A Norman window has the shape of a rectangle surmounted by a semi-circle (see picture). Such a window is to be built with a perimeter of 30 feet. What is the radius of the semi-circle that will allow the most light to pass through the window?



4. (10 points) The graph given below is of the *derivative* of f.



Use this graph of f' to sketch a graph of the function f making sure that all of the following are clear on your graph: extrema, inflection points, increase, decrease, concavity.

1. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If the average rate of change of a differentiable function on $[0,1]$ is zero, then its instantaneous rate of change must be zero somewhere between 0 and 1.

   TRUE FALSE

   Solution: The Mean Value Theorem gives us this one.

   (b) If $f'(c) = 0$ and $f''(c) = 12$, then c is a local minimum for f.

   TRUE FALSE

   Solution: The Second Derivative Test for Extrema gives us this one.

   (c) If

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

   then f grows faster than g. TRUE **FALSE**

   Solution: This is the definition of growth rates of functions.

   (d) If f is decreasing on the interval $[-1,5]$, then the left Riemann Sum is an underestimate of the area under the curve between $x = -1$ and $x = 5$.

   TRUE **FALSE**

   Solution: Since the curve is decreasing the boxes obtained using left hand endpoints will all be over the curve, hence there is too much area in our estimate making it an over estimate.

   (e) If f and g are increasing on an interval I, then $f + g$ is increasing on I.

   TRUE FALSE

   Solution: f and g increasing on I gives us that $f' > 0$ and $g' > 0$ on I, hence

   $$(f+g)' = f' + g' > 0 \quad \text{on } I \qquad \Longrightarrow \quad f + g \text{ is increasing on } I.$$

2. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

(a) Evaluate the indefinite integral

$$\int \left(e^{2x} - \frac{3}{\sqrt{1-x^2}} - 5\sin(x) \right) dx.$$

Solution:

$$\boxed{\frac{1}{2}e^{2x} - 3\sin^{-1}x + 5\cos x + c}$$

These are guess and check, so to check the guess:

$$\frac{d}{dx}\left(\frac{1}{2}e^{2x} - 3\sin^{-1}x + 5\cos x + c\right) = \frac{1}{2}\cdot 2e^{2x} - 3\cdot\frac{1}{\sqrt{1-x^2}} + 5(-\sin x) + 0$$

(b) Use L'Hospital's Rule to find the limit:

$$\lim_{x\to 0}\left(\frac{\cos x - 1}{3x^2}\right)$$

NOTE: work will be checked for the use of L'Hospital's Rule so label this work on your scratch paper.

Solution: First, a check to see that L'Hospital's Rule applies: $\cos 0 - 1 = 1 - 1 = 0$ and the denominator is clearly zero as well, so it does apply.

$$\lim_{x\to 0}\left(\frac{\cos x - 1}{3x^2}\right) = \lim_{x\to 0}\left(\frac{-\sin x}{6x}\right) = -\frac{1}{6}\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = -\frac{1}{6}\cdot 1 = \boxed{-\frac{1}{6}}$$

(c) Suppose

$$\int_7^2 f(x)dx = -7 \quad \text{and} \quad \int_2^{12} f(x)dx = 5.$$

Evaluate $\int_7^{12} f(x)dx$

Solution:

$$\int_2^{12} f(x)dx = \int_2^7 f(x)dx + \int_7^{12} f(x)dx = -\int_7^2 f(x)dx + \int_7^{12} f(x)dx$$

Hence, we get

$$5 = -(-7) + \int_7^{12} f(x)dx \implies \int_7^{12} f(x)dx = 5 - 7 = \boxed{-2}.$$

(d) Find the linear approximation to $f(x) = \sqrt[3]{x}$ at $a = 8$.

Solution: Since this is just asking for the equation of the tangent line written as $L(x) = f(8) + f'(8)(x-8)$, we will need to compute $f(8)$ and $f'(8)$. Since $f'(x) = \frac{1}{3}x^{-2/3}$, we get

$$f(8) = \sqrt[3]{8} = 2 \quad \text{and} \quad f'(8) = \frac{1}{3}(8^{-2/3}) = \frac{1}{12}.$$

$$\boxed{L(x) = 2 + \frac{1}{12}(x-8)}$$

(e) Given that the acceleration of a particle is

$$a(t) = 3t + 5 \text{ m/sec}^2,$$

find the velocity of the particle if the velocity at 2 seconds is 5 m/sec.

Solution: This is a differential equation, since acceleration is the derivative of velocity, we need the antiderivative of the acceleration.

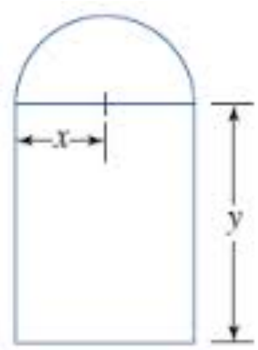
$$v(t) = \frac{3}{2}t^2 + 5t + c$$

Since we are given $v(2) = 5$, we can solve to find the c.

$$v(2) = \frac{3}{2}(2^2) + 5 \cdot 2 + c = 16 + c = 5 \quad \Longrightarrow \quad c = -11$$

$$\boxed{v(t) = \frac{3}{2}t^2 + 5t - 11}$$

3. (10 points) A Norman window has the shape of a rectangle surmounted by a semi-circle (see picture). Such a window is to be built with a perimeter of 30 feet. What is the radius of the semi-circle that will allow the most light to pass through the window?



Solution: The perimeter is half circle and 3 sides of the rectangle for a total of 30 feet:

$$P = \pi x + 2y + 2x = 30 \implies 2y = 30 - \pi x - 2x.$$

Since we need to have a window, we can see that $0 < x < \dfrac{30}{\pi+2}$. The area is that of a half circle plus the rectangle:

$$A = 2xy + \frac{1}{2}\pi x^2.$$

We solved the perimeter constraint to get the relationship between the two variables. Putting this information into the area function, we need to maximize:

$$A(x) = x(30 - \pi x - 2x) + \frac{\pi}{2}x^2 = 30x - \pi x^2 - 2x^2 + \frac{\pi}{2}x^2 = 30x - 2x^2 - \frac{\pi}{2}x^2.$$

To maximize, we take the derivative and set it equal to zero to find the critical points.

$$A'(x) = 30 - 4x - \pi x = 30 - (4+\pi)x,$$

$$A' = 0 \implies 30 - (4+\pi)x = 0 \implies x = \frac{30}{4+\pi}.$$

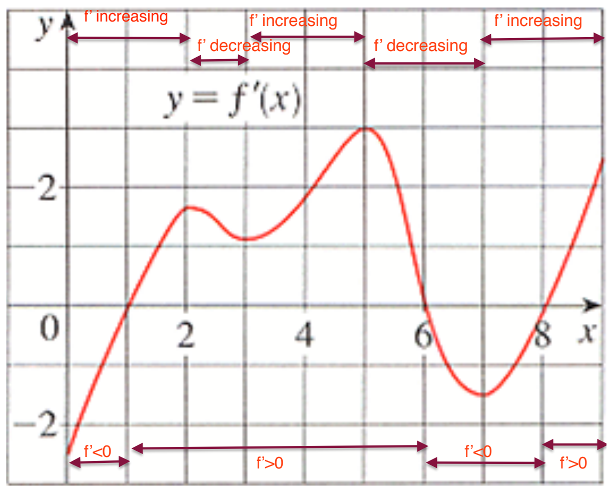
Finally, we need to verify that this gives us a maximum. I have chosen to use the second derivative test.

$$A''(x) = -(4+\pi) < 0 \quad \text{for all } x,$$

so our critical point is a maximum. The radius that allows the most light is

$$\boxed{\frac{30}{4+\pi} \text{ ft.}}$$

4. (10 points) The graph given below is of the *derivative* of f.



Use this graph of f' to sketch a graph of the function f making sure that all of the following are clear on your graph: extrema, inflection points, increase, decrease, concavity. (NOTE: all red marks and writing that are not part of the graph of f' are part of the solution and were not given.)

Solution: Recall that when f' is increasing $f'' > 0$ so the graph of f is concave up; similarly f' decreasing means the graph of f is concave down. Inflection points happen when the graph of f changes concavity, this is when the sign of f'' changes which happens when f' changes between increasing and decreasing. So the inflection points of f are the extrema of f'. The critical points of f' determine extrema of f if the sign of f' changes as we cross the critical point. Finally, $f' > 0$ tells us f is increasing and $f' < 0$ tells us f is decreasing. I have marked the relevant information on the graph of f' and indicated the effects on f below the x-axis of my picture.