MAT 270

Exam 2

SAMPLE

NOTE: This is only a sample to give an idea of form and length it is not meant to indicate exact content of your exam. The actual material covered on each exam varies from semester to semester.

NOTE: only the boxed portion of the short answer questions is the required answer that was graded, the rest of the solution is provided for explanation. For True/False questions an explanation was added which was not required.

1. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) Suppose
   $$s(t) = 3t^3 - 2t^2 + 5t$$
   gives the position, in meters, of a particle after t seconds. Find the velocity of the particle after 3 seconds.

   (b) Suppose $f(7) = 3, g(7) = 5, f'(3) = 4, g'(5) = 2, f'(7) = 11$, and $g'(7) = 6$.
   Then for $H(x) = \dfrac{f(x)}{g(x)}$, $H'(8) =$

   (c) For $f(x) = x^{\cos(x)}$, find $f'(x)$.

   (d) If a spherical snowball melts so that its surface area decreases at a rate of 3 cm^2/min, find the rate at which the radius is changing when the radius is 5 cm. The surface area of a sphere is $A = 4\pi r^2$.

   (e) For $y = e^x \ln x$, find y''.

2. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If $f(x) = 3 + 2x + e^x$, then using $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$ gives $(f^{-1})'(4) = \dfrac{1}{3}$.
   TRUE FALSE

   (b) If $f'(c) = 0$, then f has an extreme value at c. TRUE FALSE

   (c) If $g(x) = x^x$, then $g'(x) = x \cdot x^{x-1}$. TRUE FALSE

   (d) If $f(x) = \cos x$, then $f'(x) = \sin x$. TRUE FALSE

   (e) $\displaystyle\lim_{t \to 0} \frac{\sin 3t}{3t} = 1$. TRUE FALSE

3. (10 points) Suppose the tangent line of the function g at $x = 2$ is $y = -2x + 7$ and the tangent line of the function f at $x = 3$ is $y = 2x - 2$. What is the tangent line of the function $f \circ g$ at $x = 2$?

4. (10 points) Given that f and its derivatives are:

$$f(x) = 7 - 9x - 3x^2 + 3x^3, \qquad f'(x) = 3(x-3)(x+1), \quad \text{and} \quad f''(x) = 6(3x-1),$$

find the following and be sure to give your reasons.

(a) all critical point(s) of f

(b) all inflection point(s) of f

(c) all intervals on which f is increasing

(d) all intervals on which f is concave down

1. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) Suppose
   $$s(t) = 3t^3 - 2t^2 + 5t$$
   gives the position, in meters, of a particle after t seconds. Find the velocity of the particle after 3 seconds.

   Solution:
   $$v(t) = s'(t) = 9t^2 - 4t + 5 \quad \Longrightarrow \quad v(3) = 9(9) - 4(3) + 5 = \boxed{74\,\text{m/sec}}$$

   (b) Suppose $f(7) = 3, g(7) = 5, f'(3) = 4, g'(5) = 2, f'(7) = 11$, and $g'(7) = 6$.

   Then for $H(x) = \dfrac{f(x)}{g(x)}$, $H'(7) =$

   Solution:
   $$H'(7) = \frac{g(7)f'(7) - f(7)g'(7)}{[g(7)]^2} = \frac{5(11) - 3(6)}{5^2} = \boxed{\frac{37}{25}}$$

   (c) For $f(x) = x^{\cos(x)}$, find $f'(x)$.

   Solution: This requires logarithmic differentiation. Begin by taking the log on both sides of the equation: $\ln f(x) = \cos x \ln x$. Taking the derivative (implicitly) with respect to x on both sides yields
   $$\frac{1}{f(x)} f'(x) = \cos x \cdot \frac{1}{x} + \ln x(-\sin x) = \frac{\cos x}{x} - \sin x \ln x.$$
   Last, solve for $f'(x)$ to get
   $$f'(x) = f(x)\left[\frac{\cos x}{x} - \sin x \ln x\right] = \boxed{f'(x) = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]}.$$

(d) If a spherical snowball melts so that its surface area decreases at a rate of 3 cm^2/min, find the rate at which the radius is changing when the radius is 5 cm. The surface area of a sphere is $A = 4\pi r^2$.

Solution: We are told $\frac{dA}{dt} = -3$ and asked to find $\frac{dr}{dt}$ when $r = 5$. Begin by taking the derivative with respect to t on both sides of the given surface area formula and substituting in the known values.

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \quad \Longrightarrow \quad -3 = 8\pi(5)\frac{dr}{dt} = 40\pi\frac{dr}{dt}$$

Now, solve for the desired quantity:

$$\boxed{\frac{dr}{dt} = -\frac{3}{40\pi}\ \text{cm/min}}.$$

(e) For $y = e^x \ln x$, find y''.

Solution:

$$\begin{aligned} y' &= e^x \cdot \frac{1}{x} + e^x \ln x = \frac{e^x}{x} + e^x \ln x \\ y'' &= \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x \\ &= \frac{e^x}{x} - \frac{e^x}{x^2} + \frac{e^x}{x} + e^x \ln x \\ &= \boxed{\frac{2e^x}{x} - \frac{e^x}{x^2} + e^x \ln x} \end{aligned}$$

2. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If $f(x) = 3 + 2x + e^x$, then using $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$ gives $(f^{-1})'(4) = \dfrac{1}{3}$.

   $\boxed{\text{TRUE}}$ FALSE

   Solution: Using the formula provided, $(f^{-1})'(4) = \dfrac{1}{f'(f^{-1}(4))}$, we see that we need $f^{-1}(4)$. By observation we can see that $f(0) = 3 + 2(0) + e^0 = 3 + 0 + 1 = 4$, hence $f^{-1}(4) = 0$. Now we have

   $$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)}.$$

   Finding f' and substituting we get

   $$f'(x) = 2 + e^x \quad \Longrightarrow \quad f'(0) = 2 + e^0 = 2 + 1 = 3 \quad \Longrightarrow \quad \boxed{(f^{-1})'(4) = \frac{1}{3}}.$$

   (b) If $f'(c) = 0$, then f has an extreme value at c. TRUE $\boxed{\text{FALSE}}$

   Solution: Example: If $f(x) = x^3$, then $f'(0) = 0$, but the function has neither a maximum nor a minimum at zero.

   (c) If $g(x) = x^x$, then $g'(x) = x \cdot x^{x-1}$. TRUE $\boxed{\text{FALSE}}$

   Solution: This derivative must be found using logarithmic differentiation. For $y = x^x$, we have

   $$\begin{aligned} \ln y &= \ln x^x = x \ln x \\ \frac{1}{y} y' &= x \cdot \frac{1}{x} + (\ln x) \cdot 1 = 1 + \ln x \\ y' &= y(1 + \ln x) = \boxed{x^x(1 + \ln x)} \end{aligned}$$

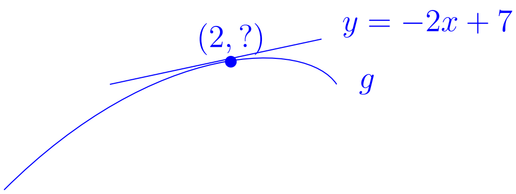
   (d) If $f(x) = \cos x$, then $f'(x) = \sin x$. TRUE $\boxed{\text{FALSE}}$

   Solution: $f'(x) = -\sin x$

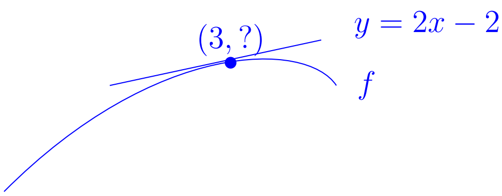
   (e) $\displaystyle\lim_{t \to 0} \frac{\sin 3t}{3t} = 1$. $\boxed{\text{TRUE}}$ FALSE

3. (10 points) Suppose the tangent line of the function g at $x = 2$ is $y = -2x + 7$ and the tangent line of the function f at $x = 3$ is $y = 2x - 2$. What is the tangent line of the function $f \circ g$ at $x = 2$?

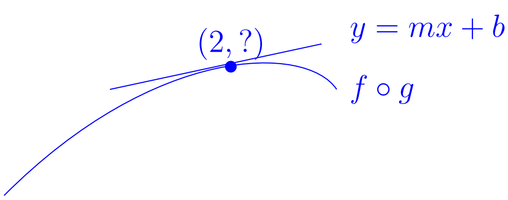
Solution:



Since the tangent line touches the curve at the point, we know the tangent goes through the point $(2, g(2))$ which is the same as the point $(2, y = -2(2) + 7)$. From this we get $g(2) = 2(2) + 7 = 3$. We also know that the slope of the tangent line at the point is the derivative of the function evaluated at the x value, so $g'(2) = -2$.



Repeating this for the function f, we get $f(3) = 2(3) - 2 = 4$ and $f'(3) = 2$.



We have

$$f \circ g(2) = f(g(2)) = f(3) = 4$$

and, using the chain rule,

$$(f \circ g)'(2) = f'(g(2)) \cdot g'(2) = f'(3)(-2) = 2(-2) = -4.$$

The tangent line to $f \circ g$ must go through the point $(2, 4)$ and it has slope -4, thus

$$\begin{aligned} y &= -4x + b \\ 4 &= -8 + b \\ 12 &= b. \end{aligned}$$

$$\boxed{y = -4x + 12}$$

4. (10 points) Given that f and its derivatives are:

$$f(x) = 7 - 9x - 3x^2 + 3x^3, \qquad f'(x) = 3(x-3)(x+1), \quad \text{and} \quad f''(x) = 6(3x-1),$$

find the following and be sure to give your reasons.

(a) all critical point(s) of f

Solution: Critical points occur where $f' = 0$ or f' does not exist. Since f' is a polynomial, it exists for all x. Setting $f' = 0$, we get

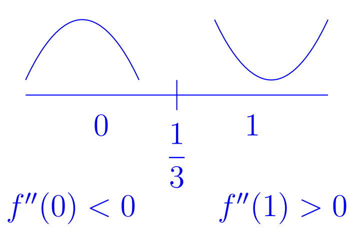
$$3(x-3)(x+1) \quad \Longrightarrow \quad \boxed{x = 3, -1}$$

(b) all inflection point(s) of f

Solution: We look for inflection points where $f'' = 0$ or f'' does not exist. Since f'' is a polynomial, it exists for all x. Setting $f'' = 0$, we get

$$6(3x-1) \quad \Longrightarrow \quad x = \frac{1}{3}.$$

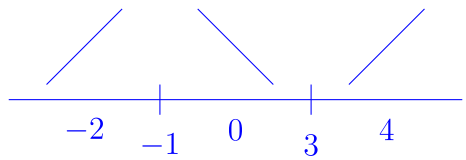
The only possible inflection point is $\frac{1}{3}$, so we test on either side of if to see if the concavity changes, that would be a change in the sign of f''.



Since the concavity changes, $\boxed{x = \frac{1}{3}}$ is an inflection point.

(c) all intervals on which f is increasing

Solution: Here we test points on either side of (and between) all critical points,



$f'(-2) > 0 \qquad f'(0) < 0 \qquad f'(4) > 0$

Hence f is increasing on $\boxed{(-\infty, -1) \cup (3, \infty)}$

(d) all intervals on which f is concave down

Solution: Based on the work in part (b), $\boxed{\left(-\infty, \frac{1}{3}\right)}$