MAT 270

Exam 1

SAMPLE

NOTE: This is only a sample to give an idea of form and length it is not meant to indicate exact content of your exam. The actual material covered on each exam varies from semester to semester.

NOTE: only the boxed portion of the short answer questions is the required answer that was graded, the rest of the solution is provided for explanation. For True/False questions an explanation was added which was not required.

1. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) Evaluate the limit *algebraically* or state that it does not exist.

   $$\lim_{x\to 7} \frac{x^2-25}{2x^2-4x-30}$$

   (b) Find the value of c that makes the function g continuous on $(-\infty,\infty)$.

   $$g(x) = \begin{cases} x^3+cx+6, & x \le -2 \\ cx^2-2x, & x > -2 \end{cases}$$

   (c) Find the limit or state that it does not exist;

   $$\lim_{x\to\infty} \frac{3}{e^x+4}.$$

   (d) Consider the function $f(x) = \dfrac{x^3-4x}{x^2+2x-3}$.

   - the number of vertical asymptotes of f is ________
   - the number of horizontal asymptotes of f is ________

   (e) Find the limit or state that it does not exist:

   $$\lim_{x\to 0} \frac{\sqrt{3x+5}-\sqrt{5}}{2x} =$$

2. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If f is continuous on the interval $[-1, 2]$, $f(-1) = 4$, and $f(2) = 6$, then there exists a number c between -1 and 2 such that $f(c) = 0$.

   TRUE FALSE

   (b) $$\lim_{x\to 1}\left(\frac{x-1}{\ln x}\right) = \lim_{x\to 1}\left((x-1)\cdot\frac{1}{\ln x}\right) = \lim_{x\to 1}(x-1)\cdot\lim_{x\to 1}\frac{1}{\ln x} = 0\cdot\lim_{x\to 1}\frac{1}{\ln x} = 0$$

   TRUE FALSE

   (c) If $f(a)$ is not defined, then $\lim_{x\to a} f(x)$ does not exist. TRUE FALSE

   (d) If $R(x)$ is a polynomial, then $\lim_{x\to a} R(x) = R(a)$. TRUE FALSE

   (e) If $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then f is continuous at $x = a$. TRUE FALSE

3. Use the limit definition of a derivative to find the derivative of

$$f(x) = \frac{5}{x-3}.$$

4. (a) Given $\lim_{x\to -2}(2x+9) = 5$. Find the values of x for which $2x+9$ is within distance $\frac{1}{10}$ of 5.

(b) Use the ϵ-δ definition of limit to prove $\lim_{x\to -2}(2x+9) = 5$.

1. (20 points) The parts of this problem are independent. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) Evaluate the limit *algebraically* or state that it does not exist.

$$\lim_{x\to 7} \frac{x^2-25}{2x^2-4x-30}$$

Solution:

$$\lim_{x\to 7} \frac{x^2-25}{2x^2-4x-30} = \lim_{x\to 7} \frac{(x-5)(x+5)}{2(x^2-2x-15)} = \lim_{x\to 7} \frac{(x-5)(x+5)}{2(x-5)(x+3)}$$
$$= \lim_{x\to 7} \frac{(x+5)}{2(x+3)} = \frac{7+5}{2(7+3)} = \frac{12}{20} = \boxed{\frac{3}{5}}$$

   (b) Find the value of c that makes the function g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^3+cx+6, & x \le -2 \\ cx^2-2x, & x > -2 \end{cases}$$

Solution: Clearly, the function is continuous everywhere except possibly at -2 since the pieces are polynomials. We need for the limit at -2 to exist. We begin by taking the limit from each side.

$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} (cx^2-2x) = c(-2)^2 - 2(-2) = 4c+4$$
$$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} (x^3+cx+6) = (-2)^3 + c(-2) + 6 = -8-2c+6 = -2c-2$$

In order for the limit to exist at -2 the limits from each side must match, so

$$4c+4 = -2c-2 \implies 6c = -6 \implies c = -1.$$

This gives

$$\lim_{x\to -2} g(x) = 4(-1)+4 = 0.$$

Finally, we need to make sure that $g(-2) = \lim_{x\to -2} g(x)$, so

$$g(-2) = (-2)^3 - (-2) + 6 = -8+2+6 = 0 = \lim_{x\to -2} g(x).$$

Hence, $\boxed{c=-1}$.

(c) Find the limit or state that it does not exist;

$$\lim_{x\to\infty} \frac{3}{e^x+4}.$$

Solution:

$$\lim_{x\to\infty} \frac{3}{e^x+4} = 3\lim_{x\to\infty} \frac{1}{e^x+4}.$$

As $x \to \infty$, e^x also grows arbitrarily large. Thus by the little-big rule, 1 over big goes to little. This limit is $\boxed{0}$.

(d) Consider the function $f(x) = \dfrac{x^3-4x}{x^2+2x-3}$.

- the number of vertical asymptotes of f is $\boxed{2}$
- the number of horizontal asymptotes of f is $\boxed{0}$

Solution: The degree in the numerator is larger than the degree in the denominator, so there are no horizontal asymptotes.

$$f(x) = \frac{x^3-4x}{x^2+2x-3} = \frac{x(x-2)(x+2)}{(x+3)(x-1)}$$

From this form, we can see there will be two vertical asymptotes.

(e) Find the limit or state that it does not exist:

$$\lim_{x\to 0} \frac{\sqrt{3x+5}-\sqrt{5}}{2x} =$$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{3x+5}-\sqrt{5}}{2x} &= \lim_{x\to 0} \frac{\sqrt{3x+5}-\sqrt{5}}{2x} \cdot \frac{\sqrt{3x+5}+\sqrt{5}}{\sqrt{3x+5}+\sqrt{5}} \\
&= \lim_{x\to 0} \frac{3x+5-5}{2x(\sqrt{3x+5}+\sqrt{5})} = \lim_{x\to 0} \frac{3x}{2x(\sqrt{3x+5}+\sqrt{5})} \\
&= \lim_{x\to 0} \frac{3}{2(\sqrt{3x+5}+\sqrt{5})} = \frac{3}{2(\sqrt{3(0)+5}+\sqrt{5})} = \boxed{\frac{3}{4\sqrt{5}}}
\end{aligned}$$

2. (10 points) True/False, circle your choice. In this problem, only the answer will be graded; you do **not** need to show any work!

   (a) If f is continuous on the interval $[-1, 2]$, $f(-1) = 4$, and $f(2) = 6$, then there exists a number c between -1 and 2 such that $f(c) = 0$.

   TRUE [FALSE]

   Solution: While the function satisfies the hypothesis of the IVT, you can only conclude that you can achieve any value between $f(-1)$ and $f(2)$. Since 0 is not between 4 and 6, you can make no conclusion about the function taking on the value 0.

   (b) $\lim_{x\to 1}\left(\frac{x-1}{\ln x}\right) = \lim_{x\to 1}\left((x-1)\cdot\frac{1}{\ln x}\right) = \lim_{x\to 1}(x-1)\cdot\lim_{x\to 1}\frac{1}{\ln x} = 0\cdot\lim_{x\to 1}\frac{1}{\ln x} = 0$

   TRUE [FALSE]

   Solution: The limit rules apply if every limit used exists. Since $\lim_{x\to 1}\frac{1}{\ln x} = \infty$, this limit does not exist and we cannot apply the rule.

   (c) If $f(a)$ is not defined, then $\lim_{x\to a} f(x)$ does not exist. TRUE [FALSE]

   Solution: This follows directly from the definition of limit, the actual function value has no bearing on the limit.

   (d) If $R(x)$ is a polynomial, then $\lim_{x\to a} R(x) = R(a)$. [TRUE] FALSE

   (e) If $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then f is continuous at $x = a$. [TRUE] FALSE

   Solution: $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then it is $f'(a)$. If a function is differentiable at a then it is continuous at a.

3. Use the limit definition of a derivative to find the derivative of

$$f(x) = \frac{5}{x-3}.$$

Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\frac{5}{x+h-3} - \frac{5}{x-3}}{h} \\
&= \lim_{h\to 0} \left[\frac{1}{h} \left(\frac{5}{x+h-3} - \frac{5}{x-3} \right) \right] \\
&= \lim_{h\to 0} \left[\frac{1}{h} \left(\frac{5(x-3) - 5(x+h-3)}{(x+h-3)(x-3)} \right) \right] \\
&= \lim_{h\to 0} \left[\frac{1}{h} \left(\frac{5x - 15 - 5x - 5h + 15)}{(x+h-3)(x-3)} \right) \right] \\
&= \lim_{h\to 0} \left[\frac{1}{h} \left(\frac{-5h}{(x+h-3)(x-3)} \right) \right] \\
&= \lim_{h\to 0} \left(\frac{-5}{(x+h-3)(x-3)} \right) \\
&= \frac{-5}{(x+0-3)(x-3)} = \frac{-5}{(x-3)^2}
\end{aligned}$$

The limit exists, so

$$\boxed{f'(x) = \frac{-5}{(x-3)^2}}.$$

4. (a) Given $\lim_{x \to -2}(2x+9) = 5$. Find the values of x for which $2x+9$ is within distance $\frac{1}{10}$ of 5.

Solution: Since we want $2x+9$ is within distance $\frac{1}{10}$ of 5, we need $|2x+9-5| < 0.1$. To find the values of x, we note that we are looking near -2 and manipulate $|2x+9-5|$ until we see some expression involving $|x-(-2)| = |x+2|$.

$$|2x+9-5| = |2x+4| = |2(x+2)| = 2|x+2|$$

We now have $|2x+9-5| = 2|x+2| < 0.1$ which we can now solve for x.

$$2|x+2| < 0.1$$
$$|x+2| < 0.05$$
$$-0.05 < x+2 < 0.05$$
$$\boxed{-2.05 < x < -1.95}$$

(b) Use the ϵ-δ definition of limit to prove $\lim_{x \to -2}(2x+9) = 5$.

Solution:
PROOF: Let $\epsilon > 0$ be given. Then for $\delta = \frac{\epsilon}{2}$. whenever $|x+2| < \delta$, we get

$$|2x+9-5| = |2x+4| = |2(x+2)| = 2|x+2| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Hence $\lim_{x \to -2}(2x+9) = 5$.

Note: Since the entire proof is the answer, it was not appropriate to "circle" the answer here.