MAT 267: Calculus III For Engineers
Final Review

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1. Find an equation $\mathbf{r}(t)$ for the line that passes through the origin at $t = 3$ and is orthogonal to the plane $x + y - 3z = 1$.

2. Find an equation of the plane through the points $(2, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 8)$.

3. Find an equation $\mathbf{r}(t)$ for the curve of intersection between the sphere $x^2 + y^2 + z^2 = 9$ and the plane $y = 2$.

4. Find the unit tangent vector to the curve $\mathbf{r}(t) = \langle t^3, e^{t-1}, 2\ln(t) \rangle$ at $t = 1$.

5. Find an equation for the tangent line of $\mathbf{r}(t) = \langle 2t + 1, t^2 + 8t - 1, \sin(4t) \rangle$ at $t = 0$.

6. Find the arc length of the curve $\mathbf{r}(t) = \langle 1, t^2, t^3 \rangle$ from $0 \leq t \leq 1$.

7. A particle has a position vector $\mathbf{r}(t) = \langle te^{-t}, t^3 - 2t, e^{-4t} \rangle$.
   Evaluate each of the following at $t = 0$.

   (a) the velocity vector, (b) the acceleration vector, (c) the speed

8. A particle with mass $m = 0.2$ kg has a force vector $\mathbf{F}(t) = \langle 1, t, t^3 \rangle$ N acting on it. At $t = 0$, its velocity is $\mathbf{v}(0) = \langle 1, -2, 1 \rangle$ m/s. Evaluate each of the following at $t = 1$.

   (a) the acceleration vector, (b) the velocity vector

9. Find the equation of the tangent plane for the function $f(x, y) = x^2 y + \frac{y^2}{x}$ at the point $(1, 3)$.

10. Let $z = \sqrt{2x + y}$, $x = uv^2$ and $y = u\sin(v)$. Evaluate $\partial z/\partial u$ and $\partial z/\partial v$ when $(u, v) = (1, \pi)$.

11. Let $f(x, y) = x^4 + x^2 y^3$. Evaluate the following at the point $(1, 2)$.

    a) the gradient vector
    b) the direction (unit vector) of steepest ascent
    c) the directional derivative along $\mathbf{v} = \langle 3, -1 \rangle$

12. Find the unit normal vector to the implicit surface $xz^3 + e^{yz} = 9$ at the point $(1, 0, 2)$.

13. Find and classify all critical points of the function $f(x, y) = x^2 - xy + y^2 - 9x + 6y$.

14. Find the point on the plane $x + y + z = 2$ that is closest to the point $(3, 0, 1)$.

15. Evaluate the double integral by reversing the order of integration:

$$\int_0^1 \int_{4y}^4 e^{x^2} \, dx \, dy$$

16. Evaluate $\iint_D y^2 \, dA$ where D is the inside of the triangle with vertices $(0,0)$, $(2,0)$, and $(2,4)$.

17. Evaluate the double integral by switching to polar coordinates.

$$\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx.$$

18. Find the volume of the solid bounded by the paraboloids $z = 3(x^2+y^2)$ and $z = 4 - x^2 - y^2$.

19. Find the volume of the region bounded by the surfaces $y = x^2$, $z = 0$ and $y + z = 1$.

20. Evaluate the triple integral $\iiint_E xz \, dV$ where E is the part of the ball $x^2 + y^2 + z^2 \leq 4$ in the first octant.

21. Evaluate the line integral $\int_C xy \, ds$ where C is the line segment from $(0,0)$ to $(2,3)$.

22. Find the work done by the force field $\mathbf{F}(x,y) = \langle 1, x \rangle$ on a particle that moves from $(5,0)$ to $(0,5)$ along the circle $x^2 + y^2 = 25$.

23. Define the vector field $\mathbf{F}(x,y,z) = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$.

    a) Prove that $\mathbf{F}$ is conservative.
    b) Find a potential function f such that $\nabla f = \mathbf{F}$.
    c) Use f to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve parameterized as $\mathbf{r}(t) = \langle 4t+1, t^2, t \rangle$, $0 \leq t \leq 2$.

24. Use Green's Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle x + y, xy - y\sin(y^2) \rangle$ and C is the positively oriented curve that encloses the circular sector bounded by the x-axis, the line $y = x/\sqrt{3}$ and the unit circle.

25. Evaluate the curl and divergence of the vector field $\mathbf{F} = \langle x^2 + 2yz, y^2 - z^3, 4xyz \rangle$ at the point $(-1,4,2)$.

26. Parameterize the portion of the cylinder $x^2 + z^2 = 9$ in the first octant with $y \leq 5$.

27. Find the surface area of the part of the cone $z = \sqrt{8(x^2+y^2)}$ that is under the plane $z = 6$.

28. Evaluate the surface integral $\iint_S 3x \, dS$ where S is parameterized as $S(u,v) = \langle u\cos(v), u\sin(v), v \rangle$, $0 \leq u \leq 1$ and $0 \leq v \leq \pi/4$.

29. Find the flux of the vector field $\mathbf{F} = \langle -y, x, z^2 \rangle$ through the part of the paraboloid $z = 3x^2 + 3y^2$ between $z = 0$ and $z = 1$ with upward orientation.

30. Use the divergence theorem to find the flux of the vector field $\mathbf{F} = \langle 2x, 3y, z^2 \rangle$ through the sphere $x^2 + y^2 + z^2 = 9$.

Answers

1. $\mathbf{r}(t) = \langle t-3, t-3, -3t+9 \rangle$
2. $4x + 2y + z = 8$
3. $\mathbf{r}(t) = \langle \sqrt{5}\cos(t),\ 2,\ \sqrt{5}\sin(t) \rangle$
4. $\hat{\mathbf{v}} = \frac{1}{\sqrt{14}} \langle 3, 1, 2 \rangle$
5. $\mathbf{L}(t) = \langle 1+2t, -1+8t, 4t \rangle$
6. $\frac{1}{27}(13^{3/2} - 8)$
7. a) $\mathbf{v}(0) = \langle 1, -2, -4 \rangle$ m/s

   b) $\mathbf{a}(0) = \langle -2, 0, 16 \rangle$ m/s^2

   c) $|\mathbf{v}(0)| = \sqrt{21}$ m/s
8. a) $\mathbf{a}(1) = \langle 5, 5, 5 \rangle$ m/s^2

   b) $\mathbf{v}(1) = \langle 6, 0.5, 2.25 \rangle$ m/s
9. $z = -3x + 7y - 6$
10. $\frac{\partial z}{\partial u}(1, \pi) = \frac{\pi}{\sqrt{2}}$

    $\frac{\partial z}{\partial v}(1, \pi) = \sqrt{2}\pi - \frac{1}{2\sqrt{2}\pi}$
11. a) $\nabla f(1, 2) = \langle 20, 12 \rangle$

    b) $\hat{\mathbf{u}} = \frac{1}{\sqrt{34}} \langle 5, 3 \rangle$

    c) $D_{\mathbf{v}} f(1, 2) = \frac{48}{\sqrt{10}}$
12. $\hat{\mathbf{n}} = \frac{1}{\sqrt{53}} \langle 4, 1, 6 \rangle$
13. $(4, -1)$ local minimum
14. $(\frac{7}{3}, -\frac{2}{3}, \frac{1}{3})$
15. $\frac{1}{8}(e^{16} - 1)$
16. $\frac{32}{3}$
17. $\frac{\pi}{4}(1 - e^{-4})$
18. 2π
19. $\frac{8}{15}$
20. $\frac{32}{15}$
21. $2\sqrt{13}$
22. $\frac{25\pi}{4} - 5$
23. a) $\nabla \times \mathbf{F} = \mathbf{0}$

    b) $f(x, y, z) = xe^{yz} + C$

    c) $9e^8 - 1$
24. $\frac{1}{3} - \frac{\pi}{12} - \frac{\sqrt{3}}{6} \approx -0.2171$
25. curl $\mathbf{F} = \langle 4, -24, -4 \rangle$

    div $\mathbf{F} = -10$
26. $\mathbf{r}(u, v) = \langle 3\cos(u), v, 3\sin(u) \rangle$
    $0 \le u \le \pi/2,\ \ 0 \le v \le 5$
27. $\frac{27\pi}{2}$
28. $2 - \frac{1}{\sqrt{2}}$
29. $\frac{\pi}{9}$
30. 180π