Spring 2025 - Mondays and Wednesdays 12:00-1:15 pm. Tempe - WXLRA307

MAT 494/APM 598. ANALYSIS & PARTIAL DIFFERENTIAL EQUATIONS: AN INTRODUCTION

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Aim of the course: This course can be profitably followed by undergraduate and graduate students as well. This is why it is cross-listed as MAT494/APM598. It is mainly designed as a *self-contained* introduction to some of the fundamental tools and ideas of analysis and partial differential equations.

Course prerequisites: MAT 371. A basic knowledge of the Riemann integral, including the improper Riemann integral. Some exposure to the Lebesgue integral will be ideal, but it is not strictly necessary as I plan to give a "soft" introduction to its essentials. Familiarity with the basics of partial differential equations (PDEs) is helpful, but not mandatory. A good working knowledge of advanced calculus and of the notion of partial derivative will be very helpful. Exposure to integration by parts, aka the Divergence Theorem, will not be essential, as I will provide a crash introduction, with lots of examples.

Course description: The first part of the course will be about the Fourier transform and its basic properties. I will show how this powerful tool is used to solve various problems involving constant coefficient partial differential equations - but not only! - which play a pervasive role in mathematics, physics and other applied sciences. I will start with the transport equation, the simplest PDE of order one, and present its solution via Fourier transform, first, and then by the method of characteristics. Then I will move to the wave equation, from its one-dimensional model, to the four-dimensional space-time. I will solve the Cauchy problem, prove uniqueness of the solution via monotonicity of the energy, and provide a rigorous mathematical proof of the Huygens principle. I will also solve the Cauchy problems for the heat and the Schrödinger equations and prove some beautiful intertwinings aspects of them. One of the highlights will be the solution, via Fourier transform, of the Cauchy problems for the quantum mechanics harmonic oscillator, and its alter ego, the Ornstein-Uhlenbeck operator, with its invariant Gaussian measure.

In its second part, the course will branch into different directions motivated by natural phenomena such as the Newton's law of gravitation, or Coulomb's law of electrostatic attraction. This will lead to the theory of harmonic functions, and to uncover some of their fundamental features, such as the mean-value property and the strong maximum principle. I will discuss in detail some beautiful engineering and geometric applications, such as symmetry in the overdetermined torsion problem for a beam, or why a closed boundaryless surface of constant mean curvature must necessarily be a sphere. If time permits, I will also give a crash course on fractional calculus using the heat equation.

Course notes: I will use a tablet for the lectures. Full lecture notes will be made available on Canvas after each class.