Concepts:

- Define the terms "algorithm" and "complexity of an algorithm."
- Determine whether a function is big-O, big-Omega, or big-Theta of another function using the formal definitions.
- Determine whether a function is big-O of another function using theorems such as "big-O estimate of sums and products of functions."
- Be able to compare the growth of functions commonly used in big-O estimates and be able to rank them in "increasing" or "decreasing" order.
- Determine whether a function is big-O, big-Omega, or big-Theta of another function using the limits of the quotients of the functions at infinity.
- Define the "order" of a function in rigorous and approximate terms.
- Evaluate the order of a given function.

Problems:

1. Fill in the blanks:

   (a) A function $f(x)$ is big-O of $g(x)$ if and only if ...

   (b) A function $f(x)$ is big-Ω of $g(x)$ if and only if ...

   (c) A function $f(x)$ is big-Θ of $g(x)$ if and only if ...

   (d) When $f(x)$ is big-Θ of $g(x)$, then we say that $f(x)$ is of ... $g(x)$.

   (e) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ be a polynomial function with real number coefficients $a_0, a_1, \ldots a_n \neq 0$. Then $f(x)$ is of order $x^{\square}$.

   (f) Suppose $f(x)$ is $O(f_1(x))$ and $g(x)$ is $O(g_1(x))$. Then $(f + g)(x)$ is big-O of ...

   (g) Suppose $f(x)$ is $O(f_1(x))$ and $g(x)$ is $O(g_1(x))$. Then $(f \cdot g)(x)$ is big-O of ...

2. Check whether the following statements are true or false.

   Suppose f and g are two positive real-valued functions and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

   (a) If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = C$, $C > 0$, then

   - $f(x)$ is $O(g(x))$.
   - $f(x)$ is not $O(g(x))$.
   - $f(x)$ is $\Omega(g(x))$.

- $f(x)$ is not $\Omega(g(x))$.
- $f(x)$ is of order $g(x)$.
- $f(x)$ is not of order $g(x)$.
- $f(x)$ grows faster than $g(x)$.

(b) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then

- $f(x)$ is $O(g(x))$.
- $f(x)$ is not $O(g(x))$.
- $f(x)$ is $\Omega(g(x))$.
- $f(x)$ is not $\Omega(g(x))$.
- $f(x)$ is of order $g(x)$.
- $f(x)$ is not of order $g(x)$.
- $f(x)$ grows faster than $g(x)$.

(c) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then

- $f(x)$ is $O(g(x))$.
- $f(x)$ is not $O(g(x))$.
- $f(x)$ is $\Omega(g(x))$.
- $f(x)$ is not $\Omega(g(x))$.
- $f(x)$ is of order $g(x)$.
- $f(x)$ is not of order $g(x)$.
- $f(x)$ grows faster than $g(x)$.

3. Use the definition of "$f(x)$ is $O(g(x))$" to show that $f(x) = 45x^4 + 25x^2 - 3(\log(x))^4 - 82x^2 - 78x + 10023$ is $O(x^4)$. Find witnesses C and k so that $|45x^4 + 25x^2 - 3(\log(x))^4 - 82x^2 - 78x + 10023| \leq C \cdot |x^4|$ when $x > k$.

4. Use the "big-O estimate of sums/products theorem" to give a big O estimate of $f(x) = (x^2 + 9\ln x + 5 \cdot 2^x) \cdot (x^2 + 9\ln x - 9x^3) + (-9x + 5 \cdot 2^x + 2x^7) \cdot (2 + 8 \cdot 1.0001^x - 9\ln x)$. Choose a function $g(x)$ as simple as possible with the smallest order such that $f(x) = O(g(x))$.

5. Find the least integer k such that $f(n)$ is $O(n^k)$ for each of the following functions. Show that $f(n)$ is $O(n^k)$, and then show that $f(n)$ is not $O(n^{k-1})$.

(a) $f(n) = 2n^3 + n^4(\log(n))$

(b) $f(n) = 2n^3 + (\log(n))^2$

(c) $f(n) = 2n + (\log(n))^2$

(d) $f(n) = \dfrac{n^4 + 3n^2 + 1}{2n^2 + 5}$

(e) $f(n) = 1^3 + 2^3 + \ldots + n^3$

6. Determine if each of the following functions is $O(x^2)$, $\Omega(x^2)$ or both, that is $\Theta(x^2)$.

(a) $f(x) = \log(2^x)$

(b) $f(x) = \dfrac{x^4}{2}$

(c) $f(x) = \lceil x \rceil \cdot \lfloor x \rfloor$

(d) $f(x) = x \log x$

(e) $f(x) = x^2 + 2x + 10000000$

(f) $f(x) = 12x + 11$

(g) $f(x) = 2^x$

(h) $f(x) = \log(x^x)$

7. Write the following functions in "increasing" order from the "smallest" to the "largest" order. That is $f(n)$ precedes $g(n)$ in our list if and only if $f(n)$ is $O(g(n))$.

$$n\log(n^7),\ \frac{n^5+4n}{n^3+10000},\ \sqrt{n}\log(n),\ 2^n,\ \log(5^n),\ n^3\log(n),\ \frac{n^3}{1000000},\ 70000\sqrt{n},\ 1.999999^{n+1}.$$

8. Use the limit of the quotient of $f(x)$ and $g(x)$ to decide if $f(x)$ is $O(g(x))$ for the following functions.

(a) $f(x) = 1000000x^2 + 23000000$ and $g(x) = 0.0001x^2$

(b) $f(x) = \log x$ and $g(x) = \ln x$

(c) $f(x) = \ln x$ and $g(x) = x$

(d) $f(x) = (\ln x)^2$ and $g(x) = x$

(e) $f(x) = 2^x$ and $g(x) = 1000000000x^2$

(f) $f(x) = 2^x$ and $g(x) = 2.00001^x$

(g) $f(x) = 1.99999^{x+1}$ and $g(x) = 2^x$

(h) $f(x) = x^2 \cdot 6^x$ and $g(x) = 6.00000001^x$

9. Let $f(n) = 4n + 30000$ and $g(n) = 3n^2 + 2n + 200$.

   (a) We know that $f(n)$ is big-O of $g(n)$. Find the lowest k for which $f(n) \leq g(n)$, if $n > k$, and justify your answer fully.

   (b) Sofia works for a tech company. Her supervisor wants her to write an algorithm which solves a task as efficiently as possible. First she came up with an algorithm which solved the problem in $3n^2 + 2n + 200$ operations for n inputs. Later she was able to write another algorithm which solved the same task in $4n + 30000$ operations. Which algorithm should be used? Is the second algorithm (the linear) necessarily better than the first one? Explain your reasoning.

Solutions:

1. Fill in the blanks:

   (a) A function $f(x)$ is big-O of $g(x)$ if and only if there are some constants C and k such that $|f(x)| \leq C \cdot |g(x)|$ for all $x > k$.

   (b) A function $f(x)$ is big-Ω of $g(x)$ if and only if there are some constants $C > 0$ and k such that $|f(x)| \geq C \cdot |g(x)|$ for all $x > k$.

   (c) A function $f(x)$ is big-Θ of $g(x)$ if and only if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$.

   (d) When $f(x)$ is big-Θ of $g(x)$ then, we say that $f(x)$ is of order $g(x)$.

   (e) Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ be a polynomial function with real number coefficients $a_0, a_1, \ldots, a_n \neq 0$. Then $f(x)$ is of order x^n.

   (f) Suppose $f(x)$ is $O(f_1(x))$ and $g(x)$ is $O(g_1(x))$. Then $(f+g)(x)$ is big-O of $\max(|f_1(x)|, |g_1(x)|)$.

   (g) Suppose $f(x)$ is $O(f_1(x))$ and $g(x)$ is $O(g_1(x))$. Then $(f \cdot g)(x)$ is big-O of $f_1(x) \cdot g_1(x)$.

2. Check whether the following statements are true or false.

   Suppose f and g are two positive real-valued functions and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

   (a) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = C$, $C > 0$, then

   - $f(x)$ is $O(g(x))$.

     True. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = C$, $C > 0$, then $\frac{f(x)}{g(x)} \leq C + 1$, equivalently $f(x) \leq (C+1) \cdot g(x)$, when x is sufficiently large. Hence, $f(x)$ is $O(g(x))$ by the definition of big-O.
     Note that multiplying an inequality by a positive number leaves the inequality symbol unchanged.

   - $f(x)$ is not $O(g(x))$.

     False.

   - $f(x)$ is $\Omega(g(x))$.

     True. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = C$, $C > 0$, then $\frac{f(x)}{g(x)} \geq \frac{C}{2} > 0$, equivalently $f(x) \geq \frac{C}{2} \cdot g(x)$, when x is sufficiently large. Hence, $f(x)$ is $\Omega(g(x))$ by the definition of big-Ω.

   - $f(x)$ is not $\Omega(g(x))$.

     False.

   - $f(x)$ is of order $g(x)$.

     True. Since $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$, $f(x)$ is $\Theta(g(x))$.

- $f(x)$ is not of order $g(x)$.

  False.

- $f(x)$ grows faster than $g(x)$.

  False. $f(x)$ is of the order $g(x)$. The functions f and g have comparable growth rates.

(b) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then

- $f(x)$ is $O(g(x))$.

  True. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then $\frac{f(x)}{g(x)} \leq 1$, equivalently $f(x) \leq g(x)$, when x is sufficiently large. Hence, $f(x)$ is $O(g(x))$ by the definition of big-O.

- $f(x)$ is not $O(g(x))$.

  False.

- $f(x)$ is $\Omega(g(x))$.

  False. Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, $g(x)$ grows faster than $f(x)$.

- $f(x)$ is not $\Omega(g(x))$.

  True.

- $f(x)$ is of order $g(x)$.

  False. Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, $g(x)$ grows faster than $f(x)$. Thus, $f(x)$ is not of the order $g(x)$.

- $f(x)$ is not of order $g(x)$.

  True.

- $f(x)$ grows faster than $g(x)$.

  False. $g(x)$ grows faster than $f(x)$.

(c) If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then

- $f(x)$ is $O(g(x))$.

  False. Since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, $f(x)$ grows faster than $g(x)$.

- $f(x)$ is not $O(g(x))$.

  True.

- $f(x)$ is $\Omega(g(x))$.

  True. If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then $\frac{f(x)}{g(x)} \geq 1$, equivalently $f(x) \geq g(x)$, when x is sufficiently large.

Hence, $f(x)$ is $\Omega(g(x))$ by the definition of big-Ω.

- $f(x)$ is not $\Omega(g(x))$.

  False.

- $f(x)$ is of order $g(x)$.

  False. Since $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$, $f(x)$ grows faster than $g(x)$. Thus, $f(x)$ is not of the order $g(x)$.

- $f(x)$ is not of order $g(x)$.

  True.

- $f(x)$ grows faster than $g(x)$.

  True.

3. Use the definition of "$f(x)$ is $O(g(x))$" to show that $f(x) = 45x^4 + 25x^2 - 3(\log(x))^4 - 82x^2 - 78x + 10023$ is $O(x^4)$. Find witnesses C and k so that $|45x^4+25x^2-3(\log(x))^4-82x^2-78x+10023| \le C\cdot|x^4|$ when $x > k$.

   By the triangle inequality $|A+B| \le |A|+|B|$, for any real numbers A and B, and the properties of the absolute value function,

   $|45x^4 + 25x^2 - 3(\log(x))^4 - 82x^2 - 78x + 10023| \le$

   $|45x^4| + |25x^2| + |-3(\log(x))^4| + |-82x^2| + |-78x| + 10023 =$

   $45x^4 + 25x^2 + 3(\log(x))^4 + 82x^2 + 78|x| + 10023.$

   If $x > 1$, then $\log(x) < x$ and $x < x^2 < x^4$. Furthermore, if $x > 11$, then $10023 < x^4$.

   Thus, if $x > 11$, $45x^4 + 25x^2 + 3(\log(x))^4 + 82x^2 + 78|x| + 10023 \le$

   $45x^4 + 25x^4 + 3x^4 + 82x^4 + 78x^4 + x^4 = 234x^4.$

   We have shown that, if $x > 11$, then $|f(x)| \le 243x^4$. This implies $f(x)$ is $O(x^4)$ with witnesses $C = 243$ and $k = 11$. Note that the witnesses are not unique.

4. Use the "big-O estimate of sums/products theorem" to give a big O estimate of $f(x) = (x^2+9\ln x+5\cdot 2^x)\cdot(x^2+9\ln x-9x^3)+(-9x+5\cdot 2^x+2x^7)\cdot(2+8\cdot 1.0001^x-9\ln x)$. Choose a function $g(x)$ as simple as possible with the smallest order such that $f(x) = O(g(x))$

   $$f(x) = \underbrace{\underbrace{\underbrace{(x^2+9\ln x+5\cdot 2^x)}_{O(\max(x^2,\ln x,2^x))=O(2^x)} \cdot \underbrace{(x^2+9\ln x-9x^3)}_{O(\max(x^2,\ln x,x^3))=O(x^3)}}_{=O(2^x\cdot x^3)} + \underbrace{\underbrace{(-9x+5\cdot 2^x+2x^7)}_{O(\max(x,2^x,x^7))=O(2^x)} \cdot \underbrace{(2+8\cdot 1.0001^x-9\ln x)}_{O(\max(1,1.0001^x,\ln x))=O(1.0001^x)}}_{=O(2^x\cdot 1.0001^x)=O(2.0002^x)}}_{=O(\max(2^x\cdot x^3,2.0002^x))=O(2.0002^x)}.$$

   Note that a higher base exponential function grows faster than a lower base exponential times a polynomial function.

5. Find the least integer k such that $f(n)$ is $O(n^k)$ for each of the following functions. Show that $f(n)$ is $O(n^k)$, and then show that $f(n)$ is not $O(n^{k-1})$.

   (a) $f(n) = 2n^3 + n^4 \log(n)$.

   Since $2n^3$ is $O(n^3)$ and $n^4 \log(n)$ is $O(n^5)$, $f(n) = 2n^3 + n^4 \log(n)$ is $O(n^5)$.

   Now we use limits to show that $f(n)$ is not $O(n^4)$. Since $\lim_{n\to\infty} \frac{2n^3 + n^4 \log(n)}{n^4} = \lim_{n\to\infty} \frac{2}{n} + \log n = \infty$, $f(n)$ grows faster than n^4. That is, $f(n)$ is not $O(n^4)$.

   Thus, the least integer k such that $f(n)$ is $O(n^k)$ is $k = 5$.

   (b) $f(n) = 2n^3 + (\log(n))^2$.

   Since $2n^3$ is $O(n^3)$ and $(\log(n))^2$ is $O(n)$ (why?), $f(n) = 2n^3 + (\log(n))^2$ is $O(n^3)$.

   We are going to show that $f(n)$ is not $O(n^2)$. From calculus we know that $\lim_{n\to\infty} \frac{\log(n)}{n} = 0$, thus $\lim_{n\to\infty} \frac{2n^3 + (\log(n))^2}{n^2} = \lim_{n\to\infty} 2n + \left(\frac{\log(n)}{n}\right)^2 = \infty$. Hence, $f(n)$ grows faster than n^2, which implies $f(n)$ is not $O(n^2)$.

   Thus, the least integer k such that $f(n)$ is $O(n^k)$ is $k = 3$.

   (c) $f(n) = 2n + (\log(n))^2$.

   Since $2n$ is $O(n)$ and $(\log(n))^2$ is $O(n)$ (why?), $f(n) = 2n + (\log(n))^2$ is $O(n)$.

   Since $\lim_{n\to\infty} \frac{2n + (\log(n))^2}{1} = \infty$, $f(n)$ is not $O(1)$.

   Thus, the least integer k such that $f(n)$ is $O(n^k)$ is $k = 1$.

   (d) $f(n) = \frac{n^4 + 3n^2 + 1}{2n^2 + 5}$.

   $f(n) = \frac{n^4 + 3n^2 + 1}{2n^2 + 5} \leq \frac{n^4 + 3n^4 + n^4}{2n^2} = \frac{5n^4}{2n^2} = 2.5n^2$, when $n \geq 1$. Thus, $f(n)$ is $O(n^2)$.

   $f(n)$ is not big-$O(n)$, since $\lim_{n\to\infty} \frac{\frac{n^4 + 3n^2 + 1}{2n^2 + 5}}{n} = \lim_{n\to\infty} \frac{n^4 + 3n^2 + 1}{2n^3 + 5n} = \infty$.

   Thus, the least integer k such that $f(n)$ is $O(n^k)$ is $k = 2$.

   (e) $f(n) = 1^3 + 2^3 + \ldots + n^3$.

   $f(n) = 1^3 + 2^3 + \ldots + n^3 \leq \underbrace{n^3 + n^3 + \ldots n^3}_{\text{n-times}} = n^4$. Thus, $f(n)$ is $O(n^4)$.

   In order to show that $f(n)$ is not $O(n^3)$, we will use the following formula $1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$. This formula is justified in the induction section.
   From the closed formula form $f(n) = \frac{n^2(n+1)^2}{4}$, we see that $f(n)$ is a polynomial function of degree 4, which is of order n^4.

   Thus, the least integer k such that $f(n)$ is $O(n^k)$ is $k = 4$.

6. Determine if each of the following functions is $O(x^2)$, $\Omega(x^2)$ or both, that is $\Theta(x^2)$.

(a) $f(x) = \log(2^x)$

First, notice that $f(x) = \log(2^x) = x \log 2$, which implies that $f(x)$ is of the order x. Thus, $f(x)$ is $O(x^2)$, but not $\Omega(x^2)$ and not $\Theta(x^2)$.

(b) $f(x) = \dfrac{x^4}{2}$

$f(x)$ is a polynomial function of degree 4, so $f(x)$ is of order x^4. Thus, $f(x)$ is not $O(x^2)$, but $f(x)$ is $\Omega(x^2)$ and $f(x)$ is not $\Theta(x^2)$.

(c) $f(x) = \lceil x \rceil \cdot \lfloor x \rfloor$

By the definition of ceiling and floor functions, $x \leq \lceil x \rceil < x + 1$ and $x - 1 < \lfloor x \rfloor \leq x$. Thus, if $x > 1$, $x(x-1) < \lceil x \rceil \cdot \lfloor x \rfloor < (x+1)x$, which implies that $f(x)$ is of order x^2. Thus, $f(x)$ is $O(x^2)$, $\Omega(x^2)$ and $\Theta(x^2)$.

(d) $f(x) = x \log x$

We know that $\log(x)$ is $O(x)$, and hence $x \log x$ is $O(x^2)$. $x \log x$ is not $\Omega(x^2)$, since $\lim_{x\to\infty} \dfrac{x \log x}{x^2} = \lim_{x\to\infty} \dfrac{\log x}{x} = 0$. Thus, $f(x)$ is not $\Theta(x^2)$.

(e) $f(x) = x^2 + 2x + 10000000$

$f(x)$ is a polynomial function of degree two. Hence, $f(x)$ is of order x^2. Thus, $f(x)$ is $O(x^2)$, $\Omega(x^2)$ and $\Theta(x^2)$.

(f) $f(x) = 12x + 11$

$f(x)$ is a polynomial function of degree one. Hence, $f(x)$ is of the order x, which implies $f(x)$ is $O(x^2)$, is not $\Omega(x^2)$ and is not $\Theta(x^2)$.

(g) $f(x) = 2^x$

We know that the exponential function $f(x) = 2^x$ increases faster than the quadratic function. Thus, $f(x)$ is not $O(x^2)$, it is $\Omega(x^2)$ and it is not of order x^2.

(h) $f(x) = \log(x^x)$

We know $f(x) = \log(x^x) = x \log x$. This example has been discussed in part (d).

7. Write the following functions in "increasing" order from the "smallest" to the "largest" order. That is $f(n)$ precedes $g(n)$ in our list if and only if $f(n)$ is $O(g(n))$.

$$n \log(n^7),\ \frac{n^5 + 4n}{n^3 + 10000},\ \sqrt{n}\log(n),\ 2^n,\ \log(5^n),\ n^3\log(n),\ \frac{n^3}{1000000},\ 70000\sqrt{n},\ 1.999999^{n+1}.$$

Note that $n\log(n^7) = 7n\log(n)$ and $\log(5^n) = n\log(5)$. Thus, the functions are in increasing order:

$$70000\sqrt{n},\ \log(5^n),\ \sqrt{n}\log(n),\ n\log(n^7),\ \frac{n^5 + 4n}{n^3 + 10000},\ \frac{n^3}{1000000},\ n^3\log(n),\ 1.999999^{n+1},\ 2^n.$$

8. Use the limit of the quotient of $f(x)$ and $g(x)$ to decide if $f(x)$ is $O(g(x))$ for the following functions.

(a) $f(x) = 1000000x^2 + 23000000$ and $g(x) = 0.0001x^2$

Since $\lim_{x\to\infty} \frac{1000000x^2 + 23000000}{0.0001x^2} = 10000000000$, $f(x)$ is $O(g(x))$. In fact, $f(x)$ is of the order $g(x)$.

In general, any two polynomial functions with the same degree have the same order.

(b) $f(x) = \log x$ and $g(x) = \ln x$

Using the change of the base formula, $\log x = \frac{\ln x}{\ln 10}$. Thus, $\lim_{x\to\infty} \frac{\log x}{\ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 10}}{\ln x} = \frac{1}{\ln 10}$, which implies that $f(x)$ is of order $g(x)$.
Thus, $f(x)$ is $O(g(x))$.

In general, any two logarithmic functions with base greater than one have the same order.

(c) $f(x) = \ln x$ and $g(x) = x$

By the L'Hospital Rule,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{1}{x} = 0.$$

Thus, $f(x)$ is $O(g(x))$ but $f(x)$ is not of order $g(x)$.

(d) $f(x) = (\ln x)^2$ and $g(x) = x$

By the L'Hospital Rule,

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\ln x}{x} = \frac{2}{x} = 0.$$

Thus, $f(x)$ is $O(g(x))$ but $f(x)$ is not of order $g(x)$.

Note that the logarithmic function grows so slowly that even $(\log(x))^n$ is $O(x)$ for any positive integer n.

(e) $f(x) = 2^x$ and $g(x) = 1000000000x^2$

By the L'Hospital Rule,

$$\lim_{x\to\infty} \frac{2^x}{1000000000x^2} = \lim_{x\to\infty} \frac{2^x \ln 2}{2000000000x} = \lim_{x\to\infty} \frac{2^x(\ln 2)^2}{2000000000} = \infty.$$

Thus, $f(x)$ grows faster than $g(x)$, and hence $f(x)$ is not $O(g(x))$.

In general, an exponential function with base greater than one grows faster than a polynomial function.

(f) $f(x) = 2^x$ and $g(x) = 2.00001^x$

We observe that,

$$\lim_{x\to\infty} \frac{2^x}{2.00001^x} = \lim_{x\to\infty} \left(\frac{2}{2.00001}\right)^x = 0,$$

since the base $\frac{2}{2.00001} < 1$. Thus, $f(x)$ is $O(g(x))$ and $f(x)$ is not of order $g(x)$.

In general, a higher base exponential function grows faster than a lower base exponential function.

(g) $f(x) = 1.99999^{x+1}$ and $g(x) = 2^x$

Using the laws of exponentiation,

$$\frac{1.99999^{x+1}}{2^x} = 1.99999\left(\frac{1.99999}{2}\right)^x.$$

Since the base $\dfrac{1.99999}{2} < 1$,

$$\lim_{x\to\infty} \frac{1.99999^{x+1}}{2^x} = \lim_{x\to\infty} 1.99999\left(\frac{1.99999}{2}\right)^x = 0.$$

Thus, $f(x)$ is $O(g(x))$. $f(x)$ is not of the order $g(x)$, since $g(x)$ grows faster than $f(x)$.

(h) $f(x) = x^2 \cdot 6^x$ and $g(x) = 6.00000001^x$

We observe that,

$$\lim_{x\to\infty} \frac{x^2 6^x}{6.00000001^x} = \lim_{x\to\infty} x^2\left(\frac{6}{6.00000001}\right)^x = 0,$$

since the base $\dfrac{6}{6.00000001} < 1$, and the exponential decay beats the polynomial growth. Thus, $f(x)$ is $O(g(x))$ and $f(x)$ is not of order $g(x)$.

In general, higher base exponential function grows faster than a lower base exponential function times a polynomial function.

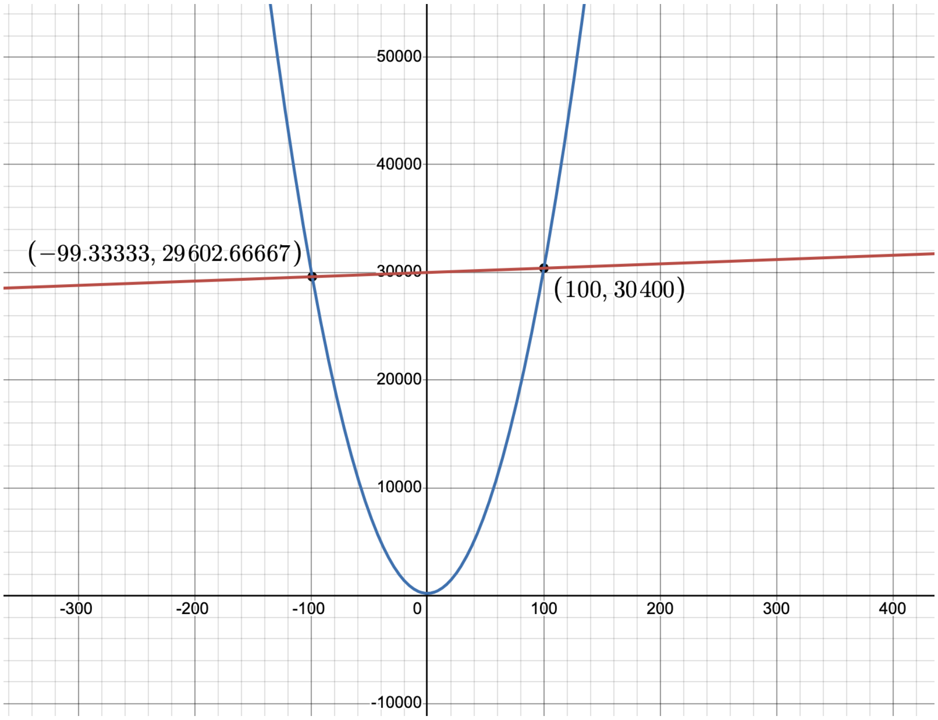
9. Let $f(n) = 4n + 30000$ and $g(n) = 3n^2 + 2n + 200$.

(a) We know that $f(n)$ is big-O of $g(n)$. Find the lowest k for which $f(n) \le g(n)$, if $n > k$, and justify your answer.

We need to find the lowest k so that $4n + 30000 \le 3n^2 + 2n + 200$ for all $n > k$. Equivalently, we need to solve the inequality $0 \le 3n^2 - 2n - 29800 = (n - 100)(3n + 298)$. This inequality is true if and only if $n \le -\dfrac{298}{3}$ or $100 \le n$, and is false if and only if $-\dfrac{298}{3} < n < 100$. Therefore, $k = 100$ is the smallest k for which $4n + 30000 \le 3n^2 + 2n + 200$ for all $n > k$.

(b) Sofia works for a tech company. Her supervisor wants her to write an algorithm which solves a task as efficiently as possible. First she came up with an algorithm which solved the problem in $3n^2 + 2n + 200$ operations for n inputs. Later she was able to write another algorithm which solved the same task in $4n + 30000$ operations. Which algorithm should be used? Is the second algorithm (the linear) necessarily better than the first one? Explain.

The second (linear) algorithm is not necessarily better. The first (quadratic) algorithm will use more operations than the linear algorithm if $n > 100$. However, if there are fewer than $n = 100$ inputs, then the first (quadratic) algorithm requires fewer operations. Thus, the answer depends on the input size n in a given situation.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