Concepts:

- Define the terms "sequence", "arithmetic sequence" and "geometric sequence."
- Convert a sequence to a closed formula for each term as a function of its position in the list.
- Be able to write recursive definitions for arithmetic, geometric sequences and for $n!$.
- Recognize sequences as geometric or arithmetic. Define such sequences directly or recursively.
- Write sums given in expanded form into sigma form.
- Evaluate sums of consecutive integers and squares, geometric sums and telescoping sums by applying summation formulas, algebraic simplifications of the sum and index shifting.

Problems:

1. Fill in the blanks:
   (a) A sequence is a function whose domain is ...
   (b) An arithmetic sequence is a (write the kind of function) ...
   (c) A geometric sequence is an (write the kind of function) ...
   (d) If $\{a_n\}_{n\in\mathbb{N}_0}$ is an arithmetic sequence with common difference d, then $d = \ldots$
   (e) If $\{a_n\}_{n\in\mathbb{N}_0}$ is a geometric sequence with quotient q, then $q = \ldots$

2. Let $\{a_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ be two sequences. Check whether the following statements are true or false.

   (a)
   $$\sum_{k=11}^{134} a_{(k-11)} = \sum_{k=0}^{123} a_k.$$

   (b)
   $$\sum_{k=0}^{123}(a_k + b_k) = \sum_{k=0}^{123} a_k + \sum_{k=0}^{123} b_k.$$

   (c)
   $$\sum_{k=0}^{123} a_k \cdot b_k = \sum_{k=0}^{123} a_k \cdot \sum_{k=0}^{123} b_k.$$

   (d)
   $$\sum_{k=0}^{123} 12 \cdot a_k = 12 \cdot \sum_{k=0}^{123} a_k.$$

3. If a sequence is arithmetic and its first two terms are 5 and 8, what is the next term?

4. If a sequence is geometric and its first two terms are 5 and 8, what is the next term?

5. Suppose $\{a_n\}_{n\in\mathbb{N}_0}$ is arithmetic and $a_5 = 7$, $a_{13} = 93$.

   (a) What is the common difference?

   (b) Find the sum, $\sum\limits_{k=5}^{100} a_k$.

   Show your work.

6. Suppose $\{a_n\}_{n\in\mathbb{N}_0}$ is geometric and $a_5 = 7$, $a_{13} = 93$.

   (a) What is the ratio? Leave your answer in radical form.

   (b) Find the sum, $\sum\limits_{k=5}^{100} a_k$.

   Show your work. Do not simplify your answer. A calculator is not permitted for this problem.

7. Use the summation formula $\sum\limits_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$ and $j = k + 3$ as an index shift to evaluate the sum. Show all your work. A calculator is not allowed for this question.

$$\sum_{k=2}^{30}(k+3)^2.$$

8. Evaluate the sum and simplify as much as possible. Show all your work. A calculator is not allowed for this question.

$$\sum_{k=2}^{45}\frac{2^{(2k+1)}}{5^k}.$$

9. Express the sum using sigma notation and evaluate it. In the sequence, each term is four more than the previous term. Show all your work. A calculator is not allowed for this question.

$$(-6) + (-2) + 2 + 6 + 10 + \ldots + 442.$$

10. Express the sum using sigma notation and evaluate it. In the sequence, each term is twice as large as the previous term. Show all your work. A calculator is not allowed for this question.

$$16 + 32 + 64 + 128 + 256 + \ldots + 2^{30}.$$

11. Find a recursive definition for the following sequences defined by the closed formulas:

    (a) $a_n = 2n$, $n \geq 0$.

    (b) $a_n = 2n + 1$, $n \geq 0$.

    (c) $a_n = 2^n$, $n \geq 0$.

    (d) $a_n = 2 - 7n$, $n \geq 0$.

    (e) $a_n = (-2) \cdot 7^n$, $n \geq 0$.

    (f) $a_n = n!$, $n \geq 0$.

    (g) $a_n = 6n$, $n \geq 0$.

    (h) $a_n = (n - 1)! \cdot 2^{(n+1)}$, $n \geq 0$.

Solutions:

1. Fill in the blanks:

   (a) A sequence is a function whose domain is restricted to the nonnegative integers.

   (b) An arithmetic sequence is a linear function whose domain is restricted to the nonnegative integers.

   (c) A geometric sequence is an exponential function whose domain is restricted to the nonnegative integers.

   (d) If $\{a_n\}_{n\in\mathbb{N}_0}$ is an arithmetic sequence with common difference d, then $d = a_{n+1} - a_n$ for all $n \in \mathbb{N}_0$.

   That is, d is the difference of two consecutive terms.

   (e) If $\{a_n\}_{n\in\mathbb{N}_0}$ is a geometric sequence with quotient q, then $q = \dfrac{a_{n+1}}{a_n}$ for all $n \in \mathbb{N}_0$.

   That is, q is the quotient of two consecutive terms.

2. Let $\{a_n\}_{n\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$ be two sequences. Check whether the following statements are true or false.

   (a)
$$\sum_{k=11}^{134} a_{(k-11)} = \sum_{k=0}^{123} a_k.$$

   It is true. Apply $j = k - 11$ index shift and replace the variable j with k in the new summation.

   (b)
$$\sum_{k=0}^{123}(a_k + b_k) = \sum_{k=0}^{123} a_k + \sum_{k=0}^{123} b_k.$$

   This is true due to the associative and commutative properties of the addition of real numbers.

   (c)
$$\sum_{k=0}^{123} a_k \cdot b_k = \sum_{k=0}^{123} a_k \cdot \sum_{k=0}^{123} b_k.$$

   This statement is false. It is simple to see that $(2+3)\cdot(4+5) = 45 \neq 23 = 2\cdot 4 + 3\cdot 5$.

   (d)
$$\sum_{k=0}^{123} 12 \cdot a_k = 12 \cdot \sum_{k=0}^{123} a_k.$$

   This is true by the distributive property of real numbers.

3. If a sequence is arithmetic and its first two terms are 5 and 8, what is the next term?

   The common difference is 3. Thus, the next term is 11.

4. If a sequence is geometric and its first two terms are 5 and 8, what is the next term?

   The common ratio is $\frac{8}{5}$. The next term is $\frac{64}{5}$.

5. Suppose $\{a_n\}_{n\in\mathbb{N}_0}$ is arithmetic and $a_5 = 7$, $a_{13} = 93$.

   (a) What is the common difference?

   By the definition of arithmetic sequence $a_{13} = a_5 + 8d$. Thus, $93 = 7 + 8d$, which implies $d = \frac{86}{8} = 10.75$.

   (b) Find the sum $\sum_{k=5}^{100} a_k$.

$$\sum_{k=5}^{100} a_k = \frac{96}{2}(a_5 + a_{100}) = 48 \cdot (a_5 + (a_5 + 95d)) = 48 \cdot (7 + 7 + 95 \cdot 10.75) = 49692.$$

6. Suppose $\{a_n\}_{n\in\mathbb{N}_0}$ is geometric and $a_5 = 7$, $a_{13} = 93$.

   (a) What is the common ratio? Leave your answer in radical form.

   By the definition of geometric sequence $a_{13} = a_5 q^8$. Thus, $93 = 7q^8$, which implies $q = \sqrt[8]{\frac{93}{7}}$.

   (b) Find the sum, $\sum_{k=5}^{100} a_k$.

   Show your work. Do not simplify your answers. A calculator is not permitted for this problem.

   By the definition geometric sequence $a_i = a_5 q^{i-5}$ for all nonnegative integers i. Now we evaluate the sum

$$\sum_{k=5}^{100} a_k = \sum_{k=5}^{100} a_5 q^{k-5}.$$

   Using the $j = k - 5$ index shift, we obtain

$$\sum_{j=0}^{95} a_5 q^j = a_5 \frac{q^{96} - 1}{q - 1} = 7 \cdot \frac{\left(\sqrt[8]{\frac{93}{7}}\right)^{93} - 1}{\sqrt[8]{\frac{93}{7}} - 1}.$$

7. Use the summation formula $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $j = k + 3$ index shift to evaluate the sum. Show all your work. A calculator is not permitted for this problem.

$$\sum_{k=2}^{30} (k+3)^2.$$

Let $j = k+3$. When $k=2$, then $j=5$, and when $k=30$, then $j=33$. Thus, $\sum_{k=2}^{30}(k+3)^2 = \sum_{k=5}^{33} j^2 = \sum_{k=1}^{33} j^2 - \sum_{j=1}^{4} j^2 = \frac{33\cdot 34\cdot 67}{6} - \frac{4\cdot 5\cdot 9}{6}$.

Note: We can use more than one method to solve the following problems. Two different methods are shown here. Students are welcome to pick any method of their choice unless specified otherwise.

8. Evaluate the sum and simplify as much as possible. Show all your work. A calculator is not permitted for this problem.

$$\sum_{k=2}^{45} \frac{2^{(2k+1)}}{5^k}.$$

(a) First solution:

$$\sum_{k=2}^{45} \frac{2^{(2k+1)}}{5^k} = \sum_{k=2}^{45} \frac{2^{(2k)}\cdot 2}{5^k} = 2\cdot\sum_{k=2}^{45}\left(\frac{4}{5}\right)^k = 2\cdot\sum_{k=0}^{45}\left(\frac{4}{5}\right)^k - 2\cdot\sum_{k=0}^{1}\left(\frac{4}{5}\right)^k$$

$$2\cdot\frac{0.8^{46}-1}{0.8-1} - 2\cdot\frac{0.8^{2}-1}{0.8-1} = 2\cdot\frac{0.8^{46}-0.8^{2}}{0.8-1}.$$

(b) Second solution:

$$\sum_{k=2}^{45} \frac{2^{(2k+1)}}{5^k} = \sum_{k=2}^{45} \frac{2^{(2k)}\cdot 2}{5^k} = 2\cdot\sum_{k=2}^{45}\left(\frac{4}{5}\right)^k$$

We will now factor out $\left(\frac{4}{5}\right)^2$ to obtain,

$$2\cdot\left(\frac{4}{5}\right)^2\cdot\sum_{k=2}^{45}\left(\frac{4}{5}\right)^{k-2}.$$

Using $j = k-2$ index shift,

$$2\cdot\left(\frac{4}{5}\right)^2\cdot\sum_{j=0}^{43}\left(\frac{4}{5}\right)^{j} = 2\cdot(0.8)^2\frac{0.8^{44}-1}{0.8-1} = 2\cdot\frac{0.8^{46}-0.8^{2}}{0.8-1}.$$

9. Express the sum using sigma notation and evaluate it. In the sequence, each term is four more than the previous term. Show all your work. A calculator is not permitted for this problem.

$$(-6)+(-2)+2+6+10+\ldots+442.$$

(a) First solution:

$$\sum_{k=0}^{112}(-6+4k) = \sum_{k=0}^{112}(-6) + 4\cdot\sum_{k=0}^{112} k = (-6)\cdot 113 + 4\cdot\frac{112\cdot 113}{2} = 24634.$$

(b) Second solution: Note that adding the first and the last term of an arithmetic sequence and multiplying this sum by the number of terms will give the double of the sum of the terms.

$$\sum_{k=0}^{112}(-6+4k) = \frac{113}{2}\cdot((-6)+442) = 24634.$$

10. Express the sum using sigma notation and evaluate it. In the sequence, each term is twice as large as the previous term. Show all your work. A calculator is not permitted for this problem.

$$16 + 32 + 64 + 128 + 256 + \ldots + 2^{30}.$$

(a) First solution:

$$\sum_{k=4}^{30} 2^k = \sum_{k=0}^{30} 2^k - \sum_{k=0}^{3} 2^k = (2^{31}-1) - (2^4-1) = 2^{31} - 2^4.$$

(b) Second solution: first factor out 2^4, then apply $j = k - 4$ index shift. Finally switch back to index k.

$$\sum_{k=4}^{30} 2^k = 2^4 \cdot \sum_{k=4}^{30} 2^{k-4} = 2^4 \sum_{j=0}^{26} 2^j = 2^4 \sum_{k=0}^{26} 2^k = 2^4 \cdot (2^{27}-1) = 2^{31} - 2^4.$$

11. Find a recursive definition for the following sequences defined by their closed formulas:

(a) $a_n = 2n$, $n \geq 0$.

$a_0 = 0;$
$a_n = a_{n-1} + 2$, $n \geq 1$.

(b) $a_n = 2n + 1$, $n \geq 0$.

$a_0 = 1;$
$a_n = a_{n-1} + 2$, $n \geq 1$.

(c) $a_n = 2^n$, $n \geq 0$.

$a_0 = 1;$
$a_n = 2 \cdot a_{n-1}$, $n \geq 1$.

(d) $a_n = 2 - 7n$ $n \geq 0$.

$a_0 = 2;$
$a_n = a_{n-1} - 7$, $n \geq 1$.

(e) $a_n = (-2) \cdot 7^n$, $n \geq 0$.

$a_0 = -2;$
$a_n = 7a_{n-1}$, $n \geq 1$.

(f) $a_n = n!$.

$a_0 = 1;$
$a_n = na_{n-1}$, $n \geq 1$.

(g) $a_n = 6n$.

$a_0 = 0$;
$a_n = a_{n-1} + 6,\ n \geq 1$.

(h) $a_n = (n-1)! \cdot 2^{(n+1)}$.

$a_1 = 4$;
$a_n = 2(n-1)a_{n-1},\ n \geq 2$.