Concepts:

- Define the concepts "function", "domain", "codomain" and "range."
- Find the domain, codomain and range of a function. Evaluate a function given by a table or a bubble diagram.
- Find the implied domain of a function given by an algebraic formula.
- Solve problems involving the floor and the ceiling functions.
- For a given function define the image and pre-image operations as functions from the power sets of the domain and codomain of the original function.
- Distinguish between the pre-image operation and the inverse function. Evaluate image and pre-image of intervals for simple functions such as the square, absolute value, floor and ceiling functions etc.
- Find the inverse of a function and the composition of two functions.
- Define the properties "one-to-one", "onto" and "bijective," "increasing" and "decreasing" functions both in intuitive (verbal) terms and by means of quantified formal statements.
- Identify functions that have certain properties on given domains and codomains.
- Prove algebraically whether a function is injective, or surjective based on the formal definition.
- Restrict the domain of a function in order to force it to be injective; restrict the codomain of a function in order to force it to be surjective.
- Prove algebraically that a function is bijective by showing that it is one-to-one and onto.

Problems:

1. Use the formal definition (with quantifiers) of function properties to fill in the blanks:

   (a) A function $f : A \to B$ is one-to-one if and only if ...

   (b) A function $f : A \to B$ is not one-to-one if and only if ...

   (c) A function $f : A \to B$ is onto if and only if ...

   (d) A function $f : A \to B$ is not onto if and only if ...

2. Which of the following statements about an arbitrary function f are correct?

   (a) The codomain and the domain are always the same.

   (b) The domain is a subset of the codomain.

   (c) A function f is not surjective if and only if there is an element y in the codomain such that $f(x) \neq y$ for all inputs x in the domain.

   (d) The codomain is the complement of the domain.

   (e) The codomain is a subset of the range.

(f) The range is a subset of the codomain.

(g) A function f is surjective if and only if for every input x in the domain there is a corresponding output y in the codomain such that $f(x) = y$.

(h) A function f is surjective if and only if its range is the same as the codomain.

(i) If a function is not surjective if and only if its range is a proper subset of the codomain.

3. Let f be a function such that $f : A \to B$, where the domain A and the codomain B are sets containing 6 elements. Which of the following statements are true?

   (a) If f is injective, then f is surjective.

   (b) If f is injective, then f is bijective.

   (c) Since the cardinality of the domain and the codomain is the same, f must be surjective.

   (d) If f is not injective, then f cannot be surjective .

   (e) If f is not surjective, then f cannot be injective.

4. Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$. Find

   (a) $f([-3, 2)) =$

   (b) $f^{-1}((0, 1)) =$

   (c) $f(f^{-1}(\{0\})) =$

   (d) $f^{-1}(f(\{3\})) =$

5. Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \sin x$. Find

   (a) $f([-\frac{\pi}{6}, 0)) =$

   (b) $f^{-1}(\{0.5\}) =$

   (c) $f(f^{-1}(\{0\})) =$

   (d) $f^{-1}(f(\{0\})) =$

6. Let$f : \mathbb{R} \to \mathbb{R}$; $f(x) = \lfloor \frac{x}{5} \rfloor$ Find.

   (a) $f([-1, 11]) =$

(b) $f^{-1}(\{16\}) =$

(c) $f(f^{-1}(\{0\})) =$

(d) $f^{-1}(f(\{0\})) =$

7. Prove that $f : \mathbb{Z} \to \mathbb{Z};\ f(n) = 3n - 5$ is one-to-one but not onto.

8. Prove that $g : \mathbb{R} \to \mathbb{R}^+ \cup \{0\};\ g(x) = (x+3)^2$ is not one-to-one but it is onto.

9. (a) Prove that $f : [-3, 5] \to [-14, 10];\ f(x) = 3x - 5$ is bijective.

   (b) Prove that $f : [-1, 1] \to [0, 2];\ f(x) = |2x + 1|$ is not bijective.

10. Find all real solutions of $2 < \lfloor 2x - 3 \rfloor \leq 5$. Give your answers in interval notation. Note that $\lfloor 2x - 3 \rfloor$ represents the floor of $2x - 3$.

11. (a) Prove that there exist real numbers x, y such that $\lfloor 2x + y \rfloor = 2 \cdot \lfloor x \rfloor + \lfloor y \rfloor$.

    (b) Disprove the statement $\lfloor 2x + y \rfloor = 2 \cdot \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.

Solutions:

1. Use the formal definition (with quantifiers) of function properties to fill in the blanks:

   (a) A function $f : A \to B$ is one-to-one if and only if $\forall x_1 \in A \forall x_2 \in A(f(x_1) = f(x_2) \to x_1 = x_2)$.

   (b) A function $f : A \to B$ is not one-to-one if and only if $\exists x_1 \in A \exists x_2 \in A((f(x_1) = f(x_2)) \wedge (x_1 \neq x_2))$.

   (c) A function $f : A \to B$ is onto if and only if $\forall y \in B \exists x \in A(f(x) = y)$.

   (d) A function $f : A \to B$ is not onto if and only if $\exists y \in B \forall x \in A(f(x) \neq y)$.

2. Which of the following statements about an arbitrary function f are correct?

   (a) The codomain and the domain are always the same.

   Incorrect. The domain and the codomain can be totally unrelated sets.

   (b) The domain is a subset of the codomain.

   Incorrect. See the answer to (a).

   (c) A function f is not surjective if and only if there is an element y in the codomain such that $f(x) \neq y$ for all inputs x in the domain.

   Correct. That is the definition of non-surjectivity. See 1 (d).

   (d) The codomain is the complement of the domain.

   Incorrect. See the answer to (a).

   (e) The codomain is a subset of the range.

   Incorrect. By definition the range is a subset of the codomain.

   (f) The range is a subset of the codomain.

   Correct.

   (g) A function f is surjective if and only if for every input x in the domain there is a corresponding output y in the codomain such that $f(x) = y$.

   Incorrect. This statement is true for any function.

   (h) A function f is surjective if and only if the range is the same as the codomain.

   Correct. This statement is equivalent to the definition of "onto."

(i) If a function is not surjective if and only if the range is a proper subset of the codomain.

Correct. This statement is equivalent to the previous statement.
The contrapositive of "if f is surjective, then the range is the same as the codomain" is "if the range is not the same as the codomain, that is the range is a proper subset of the codomain, then f is not surjective."
The contrapostive of "if the range is the same as the codomain, then the function f is surjective" is "if the function is not surjective, then the range is a proper subset of the codomain."

3. Let f be a function such that $f : A \to B$, where the domain A and the codomain B are sets containing 6 elements. Which of the following statements are true?

(a) If f is injective, then f is surjective.

True. If f is injective, the range has exactly as many elements as the domain, i.e., 6. Since the codomain also has exactly 6 elements, and the range is a subset of the codomain, the codomain has to be equal to the range.

(b) If f is injective, then f is bijective.

True. The argument in (a) shows that, in this case, if f is injective, then f is surjective. Hence, f is bijective.

(c) Since the cardinality of the domain and the codomain is the same, f must be surjective.

False. For example, if for some c in the codomain, $f(x) = c$ for all x in the domain, then f is not surjective.

(d) If f is not injective, then f cannot be surjective.

True. If f is not injective, then the cardinality of the range must be less than 6. Thus, the range is a proper subset of the codomain, and hence f is not surjective.

(e) If f is not surjective, then f cannot be injective.

True. This is the contrapositive statement of part (a).

4. Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$. Find

(a) $f([-3, 2)) = [0, 9]$.

(b) $f^{-1}((0, 1)) = (-1, 0) \cup (0, 1)$.

(c) $f(f^{-1}(\{0\})) = f(\{0\}) = \{0\}$.

(d) $f^{-1}(f(\{3\})) = f^{-1}(\{9\}) = \{-3, 3\}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \sin x$. Find

(a) $f([-\frac{\pi}{6}, 0)) = [-0.5, 0)$.

(b) $f^{-1}(\{0.5\}) = \{\frac{\pi}{6} + k \cdot 2\pi, \frac{5\pi}{6} + l \cdot 2\pi | k, l \in \mathbb{Z}\}$.

(c) $f(f^{-1}(\{0\})) = f(\{k \cdot \pi | k \in \mathbb{Z}\}) = \{0\}$.

(d) $f^{-1}(f(\{0\})) = f^{-1}(\{0\}) = \{k \cdot \pi | k \in \mathbb{Z}\}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = \lfloor \frac{x}{5} \rfloor$. Find

(a) $f([-1, 11]) = \{-1, 0, 1, 2\}$.

(b) $f^{-1}(\{16\}) = [80, 85)$.

(c) $f(f^{-1}(\{0\})) = f([0, 5)) = \{0\}$.

(d) $f^{-1}(f(\{0\})) = f^{-1}(\{0\}) = [0, 5)$.

7. Prove that $f : \mathbb{Z} \to \mathbb{Z}$; $f(n) = 3n - 5$ is one-to-one but not onto.

One-to-one: Assume n_1 and n_2 are integers such that $f(n_1) = 3n_1 - 5 = 3n_2 - 5 = f(n_2)$. Adding 5 to both sides and dividing by 3, we obtain $n_1 = n_2$. Thus, f is one-to-one according to the definition.

Not onto: We will show that there is no input n such that $f(n) = 3n - 5 = 0$.

Case 1: If $n \geq 2$, then $0 < 1 \leq 3n - 5$.

Case 2: If $n \leq 1$, then $3n - 5 \leq -2 < 0$.

In either case, $f(n) = 3n - 5 \neq 0$ for all integers n. Thus, 0 is in the codomain but not in the range of the function f, which implies that f is not onto.

8. Prove that $g : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$; $g(x) = (x + 3)^2$ is not one-to-one but it is onto.

Not one-to-one: Let $x_1 = 0$ and $x_2 = -6$. Then $g(x_1) = g(x_2) = 9$ but $x_1 = 0 \neq 6 = x_2$.

Onto: Let y be an arbitrary non-negative real number in the codomain. Define $x = \sqrt{y} - 3$, which is a real number. Then $g(x) = g(\sqrt{y} - 3) = (\sqrt{y} - 3 + 3)^2 = y$.

9. (a) Prove that $f : [-3, 5] \to [-14, 10]$; $f(x) = 3x - 5$ is bijective.

Well defined: Assume $x \in [-3, 5]$. By the definition of intervals, $-3 \leq x \leq 5$. Multiplying the inequalities by 3 and subtracting 5, we obtain $-14 \leq 3x - 5 \leq 10$. Thus, the output $f(x) = 3x - 5$ lies in the interval $[-14, 10]$ (codomain) by the definition of intervals.

One-to-one: Let $a, b \in [-3, 5]$ be such that $f(a) = f(b)$. Hence, $3a - 5 = 3b - 5$. Adding 5 to both sides and dividing by 3, we obtain $a = b$.

Onto: Let $y \in [-14, 10]$. Pick $x = \dfrac{y+5}{3}$. Then $f(x) = 3 \cdot \dfrac{y+5}{3} - 5 = y + 5 - 5 = y$. Now we will show that x is in the domain $[-3, 5]$. Since $y \in [-14, 10]$, $-14 \leq y \leq 10$ by the definition of intervals. Adding 5 to all sides of the inequality $-14 \leq y \leq 10$, and then dividing all sides by 3, we get $-3 \leq x = \dfrac{y+5}{3} \leq 5$. Hence, $x \in [-3, 5]$.

(b) Prove that $f : [-1, 1] \to [0, 2]; f(x) = |2x + 1|$ is not bijective.

In order to justify that f is not bijective, we could either show that f is not onto or not one-to-one. We show that f is **not one-to-one.** For $x_1 = -1$ and $x_2 = 0$, $f(-1) = |-1| = 1 = |1| = f(0)$. Thus, $f(x_1) = f(x_2) = 1$ but $x_1 = -1 \neq 0 = x_2$.

10. Find all real solutions of $2 < \lfloor 2x - 3 \rfloor \leq 5$. Give the answer in interval notation. Note that $\lfloor 2x - 3 \rfloor$ represents the floor of $2x - 3$.

Note that, if the floor value of a number is greater than 2, then that number must be greater or equal to 3. Further, if the floor value of a number is less than or equal to 5, then that number must be less than 6.

Thus, the inequality $2 < \lfloor 2x - 3 \rfloor \leq 5$ is equivalent to another inequality $3 \leq 2x - 3 < 6$ which does not contain the floor function. Adding 3 and dividing by 2 all sides of this inequality, we obtain $3 \leq x < 4.5$. Thus, the solution in interval notation is $[3, 4.5)$.

11. (a) Prove that there exist real numbers x, y such that $\lfloor 2x + y \rfloor = 2 \cdot \lfloor x \rfloor + \lfloor y \rfloor$.

Pick $x = 1$ and $y = 0$. Then $\lfloor 2x + y \rfloor = 2 \cdot \lfloor x \rfloor + \lfloor y \rfloor = 2$.

(b) Disprove the statement $\lfloor 2x + y \rfloor = 2 \cdot \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x, y.

Pick $x = y = \frac{1}{2}$. Then $\lfloor 2x + y \rfloor = 1$ and $2 \cdot \lfloor x \rfloor + \lfloor y \rfloor = 0$. Hence, $\lfloor 2x + y \rfloor \neq 2 \cdot \lfloor x \rfloor + \lfloor y \rfloor$ for $x = y = \frac{1}{2}$.