Concepts:

- Explain the concept of a set (a collection of distinguishable objects), and how it is different from a list of objects.
- Define: the empty set, singletons, the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers.
- Write intervals in proper interval notation. Distinguish between the set $\{a, b\}$ and the intervals $[a, b]$ or (a, b).
- Identify subset and equal relationships between sets.
- Determine the cardinality of finite sets.
- Determine the power set of a set and the power set of the power set of a set.
- Find the Cartesian product of sets.
- Find the union, intersection, symmetric difference, difference between sets, and the complement of a set.
- Use Venn diagrams to solve problems.
- Prove set identities by applying the definitions of set operators in terms of logical operators and using the corresponding laws of logic.

Problems:

1. Complete the following definitions involving sets:

   (a) A is a subset of B, $A \subseteq B$, if and only if ...

   (b) A is a proper subset of B, $A \subset B$, if and only if ...

   (c) The empty set, denoted by $\{\}$ or $\emptyset$ is a set which ...

   (d) For any real numbers a and b, $a < b$, the interval $[a, b]$ is the set of ...

   (e) The power set of a set S is $\mathcal{P}(S) = \{\ldots\}$

   (f) The Cartesian product of two sets A and B is $A \times B = \{\ldots\}$

   (g) The union of two sets A and B is $A \cup B = \{\ldots\}$

   (h) The intersection of two sets A and B is $A \cap B = \{\ldots\}$

   (i) The symmetric difference of two sets A and B is $A \triangle B = \{\ldots\}$

   (j) The difference of two sets A and B is $A \setminus B = \{\ldots\}$

   (k) The complement of a set A with respect to the universal set U is $\overline{A} = \{\ldots\}$

2. Check whether the following statements are true or false.

(a) The sets $\emptyset$ and $\{\emptyset\}$ are the same.

(b) The set $\{1, 2\}$ has two elements.

(c) The set $[1, 2]$ has two elements.

(d) The sets $\{1, 2\}$ and $\{1, 1, 2, 2, 2\}$ are the same.

(e) The set $\{1, 1, 2, 2, 2\}$ has 5 elements.

(f) $\{1, 2\} \times \emptyset = \{(1, \emptyset), (2, \emptyset)\}$.

(g) If the universal set is $[0, 3]$, then the complement of the set $[1, 2]$ is the set $\{0, 3\}$.

(h) If the universal set is $\{0, 1, 2, 3\}$, then the complement of the set $\{1, 2\}$ is the set $\{0, 3\}$.

(i) The set natural numbers is a subset of the set of rational numbers.

(j) The set of natural numbers is a proper subset of the set of rational numbers.

(k) The intersection of the set of rational numbers and the set of irrational numbers is $\{0\}$. That is, the set containing the number 0.

(l) The union of the set of rational numbers and the set of irrational numbers is the set of real numbers.

(m) If $|A| = n$, then the cardinality of the power set of A is 2^n.

(n) The Cartesian product of two sets is a commutative operation. That is, for any set A and B, $A \times B = B \times A$.

(o) The notation $\{[1, 2]\}$ denotes the set of real numbers that are between 1 and 2 including the endpoints.

3. Check whether the following statements are true or false.

(a) The empty set is an element of every set.

(b) The empty set is a subset of every set.

(c) The empty set is an element of the power set of any set.

(d) $\{1\} \in \{1, 2\}$.

(e) $\{1\} \subseteq \{1, 2\}$.

(f) $\{1\} \subset \{1, 2\}$.

(g) $\{1, 2\} \subseteq \{1, 2\}$.

(h) $\{1, 2\} \subset \{1, 2\}$.

(i) $\emptyset \in \{1, 2, 3\}$.

(j) $\emptyset \subseteq \{1, 2, 3\}$.

(k) $\emptyset \in \{\emptyset, 1, 2, 3\}$.

(l) $\{1\} \in \{1, \{1, 2\}\}$.

(m) $\{1\} \subseteq \{1, \{1, 2\}\}$.

(n) $\{1, 2\} \subseteq \{1, \{1, 2\}\}$.

4. Let $A = \{\emptyset, \{0, 1\}\}$ and $B = \{2, a\}$.

- Find $\mathcal{P}(A)$.
- Find $A \times B$.

5. Let A and B be the same sets as in the previous problem. Check whether the following statements are true or false.

(a) $\emptyset \in \mathcal{P}(A)$.

(b) $\{\emptyset\} \subseteq \mathcal{P}(A)$.

(c) $\emptyset \in A$.

(d) $\{\{0, 1\}\} \subseteq A$.

(e) $\{\{0, 1\}, 2\} \in A \times B$.

(f) $\emptyset \subseteq \mathcal{P}(A)$.

(g) $\{\emptyset\} \in \mathcal{P}(A)$.

(h) $\{0, 1\} \subseteq A$.

(i) $(\emptyset, \emptyset) \in A \times A$.

(j) $\{(\emptyset, 2)\} \subseteq A \times B$.

(k) $(\emptyset, 2) \in A \times B$.

6. Prove the following statements:

(a) $[2, 4) \cap (3, 6]$ is not the empty set.

(b) $(3, 3.999999) \subseteq (2.999999, 4)$.

7. Disprove the following statements:

   (a) $(2, 3) \subseteq (2.01, 3)$.

   (b) If A, B and C are sets and $A \cup B = A \cup C$ then $B = C$.

8. What is the complement of the following sets? The universal set is the real numbers. Give your answers in interval notation. You do not need to justify.

   (a) $[1, 2)$.

   (b) $(-\infty, -1] \cup (0, \infty)$.

9. Prove that $(-1, 2] \cap (1, 3] = (1, 2]$.

10. Express the following union and intersection as a single interval. You do not need to justify.

    (a) $\bigcup_{k=5}^{10} [-k, \frac{1}{k})$.

    (b) $\bigcap_{k=5}^{10} [-k, \frac{1}{k})$.

Solutions:

1. Complete the following definitions about sets:

   (a) A is a subset of B, $A \subseteq B$, if and only if $\forall x(x \in A \to x \in B)$. That is, if x is an element of A, then x must be in B.

   (b) A is a proper subset of B, $A \subset B$, if and only if A is a subset of B but $A \neq B$.

   (c) The empty set, denoted by $\{\}$ or $\emptyset$ is a set which has no elements.

   (d) For any real numbers a and b, $a < b$, the interval $[a, b]$ is the set $\{x | x$ is a real number such that $a \leq x \leq b\}$. That is, the continuum of real numbers between a and b including the endpoints.

   (e) The power set of a set S is $\mathcal{P}(S) = \{A | A \subseteq S\}$. That is, the set of subsets of S.

   (f) The Cartesian product of two sets A and B is $A \times B = \{(a, b) | a \in A \wedge b \in B\}$. That is, the set of ordered pairs (a, b) such that a is an element of A and b is an element of B.

   (g) The union of two sets A and B is $A \cup B = \{x | x \in A \vee x \in B\}$ That is, the set of elements x that are either in A or B or both.

   (h) The intersection of two sets A and B is $A \cap B = \{x | x \in A \wedge x \in B\}$. That is, the set of elements x that are in both sets A and B.

   (i) The symmetric difference of two sets A and B is $A \triangle B = \{x | x \in A \oplus x \in B\}$. That is, the set of elements that are contained in exactly one of the sets.

   (j) The difference of two sets A and B is $A \setminus B = \{x | x \in A \wedge x \notin B\}$. That is, the set of elements that are in A but not in B.

   (k) The complement of a set A with respect to the universal set U is $\overline{A} = \{x | x \in U \setminus A\}$. That is, the set of elements that are in U but not in A.

2. Check whether the following statements are true or false.

   (a) The sets $\emptyset$ and $\{\emptyset\}$ are the same.

   False. $\emptyset$ is the empty set and $\{\emptyset\}$ is the set containing the empty set. The set $\{\emptyset\}$ has one element while the empty set $\emptyset$ has no elements.

   (b) The set $\{1, 2\}$ has two elements.

   True. The set $\{1, 2\}$ contains the elements 1 and 2.

   (c) The set $[1, 2]$ has two elements.

   False. The set $[1, 2]$ is an interval which contains infinitely many real numbers. An interval is a continuum points. The "neighboring" points in an interval are so close to each other that they are essentially indistinguishable.

(d) The sets $\{1, 2\}$ and $\{1, 1, 2, 2, 2\}$ are the same.

True. A set is a collection of indistinguishable objects. An object can not be a member of a set more than once. In a list an object can occur multiple times but not in a set.

(e) The set $\{1, 1, 2, 2, 2\}$ has 5 elements.

False. The set $\{1, 1, 2, 2, 2\}$ has only two elements 1 and 2, since $\{1, 1, 2, 2, 2\} = \{1, 2\}$.

(f) $\{1, 2\} \times \emptyset = \{(1, \emptyset), (2, \emptyset)\}$.

False. $\{1, 2\} \times \emptyset = \emptyset$, since the empty set has no elements. Note that $\{1, 2\} \times \{\emptyset\} = \{(1, \emptyset), (2, \emptyset)\}$.

(g) If the universal set is $[0, 3]$, then the complement of the set $[1, 2]$ is the set $\{0, 3\}$.

False. If the universal set is $[0, 3]$, then the complement of the set $[1, 2]$ is $[0, 1) \cup (2, 3]$.

(h) If the universal set is $\{0, 1, 2, 3\}$, then the complement of the set $\{1, 2\}$ is the set $\{0, 3\}$.

True. The complement of the set $\{1, 2\}$ with the universal set $\{0, 1, 2, 3\}$ is $\{0, 1, 2, 3\} \setminus \{1, 2\} = \{0, 3\}$.

(i) The set of natural numbers is a subset of the set of rational numbers.

True. Every natural number is a rational according to the definition of rational numbers.

(j) The set of natural numbers is a proper subset of the set of rational numbers.

True. The set of natural numbers contains only nonnegative integers and the set of rational numbers contains proper fractions as well, such as $2/3$, $5/7$, etc., in addition to the nonnegative integers.

(k) The intersection of the set of rational numbers and the set of irrational numbers is $\{0\}$. That is, the set containing the number 0.

False. The intersection of the set of rational numbers and the set of irrational numbers is the empty set. A real number is either rational or irrational but not both.

(l) The union of the set of rational numbers and the set of irrational numbers is set of real numbers.

True. A real number is irrational if it is not rational by definition.

(m) If $|A| = n$, then the cardinality of the power set of A is 2^n.

True. It is proved or demonstrated in lecture.

(n) The Cartesian product of two sets is a commutative operation. That is, for any set A and B, $A \times B = B \times A$.

False. The Cartesian product of two sets is not a commutative operation. For example: $\{1, 2\} \times \{a\} = \{(1, a), (2, a)\}$ but $\{a\} \times \{1, 2\} = \{(a, 1), (a, 2)\}$, and $\{(1, a), (2, a)\} \neq \{(a, 1), (a, 2)\}$.

(o) The notation $\{[1, 2]\}$ denotes the set of real numbers that are between 1 and 2 including the endpoints.

False. $\{[1,2]\}$ denotes the set containing the closed interval $[1,2]$. The set $\{[1,2]\}$ has only one element, namely the closed interval $[1,2]$.

3. Check whether the following statements are true or false.

   (a) The empty set is an element of any set.

   False. The empty set is a subset of any set, but not necessarily an element. For example: $\emptyset$ is not an element of the set $\{1,2,3\}$ but it is an element of the set $\{\emptyset,1,2,3\}$.

   (b) The empty set is a subset of any set.

   True. According to the definition of subsets: for any set B, $\emptyset \subseteq B$ if and only if $\forall x(x \in \emptyset \to x \in B)$. $\forall x(x \in \emptyset \to x \in B)$ is a true statement, since the premise is false for any x.

   (c) The empty set is an element of the power set of any set.

   True. The power set of a set S is the set of subsets of S. Since the empty set is a subset of any set, the power set always contains the empty set as an element.

   (d) $\{1\} \in \{1,2\}$.

   False.$\{1\} \notin \{1,2\}$ but $1 \in \{1,2\}$.

   (e) $\{1\} \subseteq \{1,2\}$.

   True. $\{1\} \subseteq \{1,2\}$, since $1 \in \{1,2\}$.

   (f) $\{1\} \subset \{1,2\}$.

   True. $\{1\} \subset \{1,2\}$, since $\{1\} \neq \{1,2\}$ and $\{1\}$ is a subset of $\{1,2\}$.

   (g) $\{1,2\} \subseteq \{1,2\}$.

   True. Every set is a subset of itself.

   (h) $\{1,2\} \subset \{1,2\}$.

   False. For any set A, A is a subset of A but not a proper subset.

   (i) $\emptyset \in \{1,2,3\}$.

   False. The elements of the set $\{1,2,3\}$ are $1,2$ and 3.

   (j) $\emptyset \subseteq \{1,2,3\}$.

   True. The empty set is a subset of any set.

   (k) $\emptyset \in \{\emptyset,1,2,3\}$.

   True. The elements of the set $\{\emptyset,1,2,3\}$ are $\emptyset,1,2$ and 3.

(l) $\{1\} \in \{1, \{1, 2\}\}$.

False. The elements of the set $\{1, \{1, 2\}\}$ are 1 and $\{1, 2\}$.

(m) $\{1\} \subseteq \{1, \{1, 2\}\}$.

True. $\{1\} \subseteq \{1, \{1, 2\}\}$, since $1 \in \{1, \{1, 2\}\}$.

(n) $\{1, 2\} \subseteq \{1, \{1, 2\}\}$.

False. $\{1, 2\} \nsubseteq \{1, \{1, 2\}\}$, since $2 \notin \{1, \{1, 2\}\}$.

4. Let $A = \{\emptyset, \{0, 1\}\}$ and $B = \{2, a\}$.

- Find $\mathcal{P}(A)$.

  $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{0, 1\}\}, \{\emptyset, \{0, 1\}\}\}$.

- Find $A \times B$.

  $A \times B = \{(\emptyset, 2), (\emptyset, a), (\{0, 1\}, 2), (\{0, 1\}, a)\}$.

5. Let A and B be the same sets as in the previous problem. Check whether the following statements are true or false.

(a) $\emptyset \in \mathcal{P}(A)$.

True. Since the empty set is a subset of any set, the power set always contains the empty set as an element.

(b) $\{\emptyset\} \subseteq \mathcal{P}(A)$.

True. $\{\emptyset\} \subseteq \mathcal{P}(A)$, since $\emptyset \in \mathcal{P}(A)$.

(c) $\emptyset \in A$.

True. The elements of A are the empty set $\emptyset$ and the set $\{0, 1\}$.

(d) $\{\{0, 1\}\} \subseteq A$.

True. $\{\{0, 1\}\} \subseteq A$, since $\{0, 1\} \in A$.

(e) $\{\{0, 1\}, 2\} \in A \times B$.

False. The elements of $A \times B$ are ordered pairs and not sets. $(\{0, 1\}, 2)$ is an element of $A \times B$ but not $\{\{0, 1\}, 2\}$.

(f) $\emptyset \subseteq \mathcal{P}(A)$.

True. The empty set is a subset of any set.

(g) $\{\emptyset\} \in \mathcal{P}(A)$.

True. The elements of $\mathcal{P}(A)$ are $\emptyset, \{\emptyset\}, \{\{0,1\}\}$ and $\{\emptyset, \{0,1\}$.

(h) $\{0,1\} \subseteq A$.

False. Neither 0 nor 1 are elements of A.

(i) $(\emptyset, \emptyset) \in A \times A$.

True. $(\emptyset, \emptyset) \in A \times A$, since $\emptyset \in A$.

(j) $\{(\emptyset, 2)\} \subseteq A \times B$.

True. $\{(\emptyset, 2)\} \subseteq A \times B$, since $(\emptyset, 2) \in A \times B$.

(k) $(\emptyset, 2) \in A \times B$.

True. The elements of $A \times B$ are $(\emptyset, 2), (\emptyset, a), (\{0,1\}, 2)$ and $(\{0,1\}, a)$.

6. Prove the following statements:

(a) $[2,4) \cap (3,6]$ is not the empty set.

Proof: We will show that $3.5 \in [2,4)$ and $(3,6]$. Since $2 \leq 3.5 < 4$ and $3 < 3.5 \leq 6$, $3.5 \in [2,4)$ and $3.5 \in (3,6]$ by the definition of intervals. Since $3.5 \in [2,4)$ and $3.5 \in (3,6], 3.5 \in [2,4) \cap (3,6]$ by the definition of the intersection of sets. Thus, $[2,4) \cap (3,6]$ is not the empty set. $\blacksquare$

(b) $(3, 3.999999) \subseteq (2.999999, 4)$.

Proof: Assume $x \in (3, 3.999999)$. By the definition of intervals, $3 < x < 3.999999$. Since $2.999999 < 3$ and $3.999999 < 4$, $2.999999 < x < 4$. Thus, by the definition of intervals $x \in (2.999999, 4)$. We have shown that, if $x \in (3, 3.999999)$, than $x \in (2.999999, 4)$. Thus, by the definition of subsets, $(3, 3.999999) \subseteq (2.999999, 4)$. $\blacksquare$

7. Disprove the following statements:

(a) $(2,3) \subseteq (2.01, 3)$.

We will show that 2.001 is in $(2,3)$ but 2.001 in not an element of $(2.01, 3)$.
Since $2 < 2.001 < 3$, and $2.001 < 2.01$, $2.001 \in (2,3)$ but $2.001 \notin (2.01, 3)$ by the definition of intervals. Thus, we have shown that 2.1 is an element of $(2,3)$ but not an element of $(2.01, 3)$ which implies $(2,3) \nsubseteq (2.01, 3)$.

(b) If A, B and C are sets and $A \cup B = A \cup C$, then $B = C$.

We use a counter example to disprove the statement. Let $A = \{1,2\}$, $B = \{2,3\}$ and $C = \{3\}$. Then $A \cup B = A \cup C = \{1,2,3\}$ but $B = \{2,3\} \neq \{3\} = C$.

8. What is the complement of the following sets? The universal set is the real numbers. Give your answers in interval notation. You do not have to justify.

(a) $[1, 2)$.

$(-\infty, 1) \cup [2, \infty)$.

(b) $(-\infty, -1] \cup (0, \infty)$.

$(-1, 0]$.

9. Prove that $(-1, 2] \cap (1, 3] = (1, 2]$.

Proof: The proof consists of two parts. First, we will show that $(-1, 2] \cap (1, 3] \subseteq (1, 2]$, and then we justify $(1, 2] \subseteq (-1, 2] \cap (1, 3]$.

- Assume $x \in (-1, 2] \cap (1, 3]$. Then by the definition of intersection of sets, $x \in (-1, 2]$ and $x \in (1, 3]$. By the definition of intervals, $-1 < x \leq 2$ and $1 < x \leq 3$, which implies that $1 < x \leq 2$. By the definition of intervals $x \in (1, 2]$. We have shown that for all real numbers x, if $x \in (-1, 2] \cap (1, 3]$, then $x \in (1, 2]$. That is, $((-1, 2] \cap (1, 3]) \subseteq (1, 2]$ according to the definition of subsets.

- Assume $x \in (1, 2]$. By the definition of intervals $1 < x \leq 2$. Since $-1 < 1$ and $2 \leq 3$, $-1 < x \leq 2$ and $1 < x \leq 3$, which implies $x \in (-1, 2]$ and $x \in (1, 3]$. By the definition of intersection of sets, $x \in (-1, 2] \cap (1, 3]$. Thus, we have shown that $(1, 2] \subseteq (-1, 2] \cap (1, 3]$.

  Since $(-1, 2] \cap (1, 3] \subseteq (1, 2]$ and $(1, 2] \subseteq (-1, 2] \cap (1, 3]$, $(-1, 2] \cap (1, 3] = (1, 2]$. ■

10. Express the following union and intersection as a single interval. You do not need to justify.

(a) $\displaystyle\bigcup_{k=5}^{10} [-k, \tfrac{1}{k})$.

$[-10, \frac{1}{5})$.

(b) $\displaystyle\bigcap_{k=5}^{10} [-k, \tfrac{1}{k})$.

$[-5, \frac{1}{10})$.