Concepts:

- Define the terms "Theorem", "Proposition", "Lemma", "Corollary", "Axiom."
- Define the concepts of odd and even, rational and irrational numbers and use them in proofs.
- Construct direct and indirect (by contraposition or by contradiction) proofs of simple theorems for the
  - "for all",
  - "there exists",
  - "for all ..., there exists ...",
  - "there exists ..., for all ..."

  type of statements.
- Recognize and avoid common proof writing mistakes such as "assuming the conclusion" and "proof by example."

1. Complete the definitions of the following concepts:

   (a) A mathematical proof is ...

   (b) An axiom is ...

   (c) A theorem is ...

   (d) A proposition is ...

   (e) A lemma is ...

   (f) A corollary is ...

2. Prove the following statement using direct proof techniques:

   For all integers n, m and p, if $n+p$ is odd and $m+p$ is odd, then $n+m$ is even.

3. Prove the following statement using contraposition:

   For all real numbers r and s, if $r+s$ is irrational, then $2r-s$ is irrational or $2s-r$ is irrational

4. Prove the following statement by contradiction:

   For all odd integers a, b, and c, if x is a solution of the equation $ax^2+bx+c=0$, then x is irrational.

5. Find the correct proof for the following theorem:

   The sum of an odd integer x and an even integer y is odd.

   (a) Assume x is an arbitrary even and y is an arbitrary odd integer. By definition $x=2n$ and $y=2n+1$ for some integer n. Then $x+y=2n+(2n+1)=4n+1=2\cdot(2n)+1$ which is a odd number by the definition of odd.

(b) Let $x = 4$ and $y = 3$. Then $x + y = 7$. Since $7 = 2 \cdot 3 + 1$, by definition 7 is odd. Therefore the sum of an even and odd number is odd.

(c) Assume x is an arbitrary even and y is an arbitrary odd integer. By definition $x = 2n$ and $y = 2m + 1$ for some integers n, m. Then $x + y = 2n + (2m + 1) = 2(n + m) + 1$. Since $n + m$ is an integer, $x + y$ is a odd number by the definition of odd.

(d) Assume x is an arbitrary even and y is an arbitrary odd integer. By definition $x = 2n$ and $y = 2m + 1$ for arbitrary integers n, m. Then $x + y = 2n + (2m + 1) = 2(n + m) + 1$. Since $n + m$ is an integer, $x + y$ is a odd number by the definition of odd numbers.

(e) We use proof by contraposition. Assume $x + y$ is even. By the definition of even $x + y = 2n$ for some integer n. Then
Case 1: $x = 2n - 2$ and $y = 2$.
Case 2: $x = 2n - 1$ and $y = 1$.
In the above two cases, both x and y are either even or odd integers. Thus, the theorem is proved by contraposition.

(f) We use proof by contradiction to prove this theorem. Assume x is an arbitrary even, y is an arbitrary odd integer and $x + y$ is even. Then $-y$ is also an odd integer. Since $x = (x + y) + (-y)$, x is a sum of an even and an odd integer, x is odd. This contradicts the assumption that x is even.

6. Several students at Hogwarts took the OWL exam. The students had to prove the following theorem from definitions and principles. Below you can see Ron's, Harry's, Ginny's and Hermione's attempts. Which student(s) proved the statement correctly?

For all real numbers x and $y \neq 0$, if $\frac{x}{y}$ is irrational, then x is irrational or y is irrational.

- Ginny attempted a proof by contraposition: Assume $\frac{x}{y}$ is rational. By the definition of rational $\frac{x}{y} = \frac{p}{q}$ for some integers $p, q \neq 0$. Then $x = p$ and $y = q$. Since p and q are integers, x and y are rational numbers. Thus, we have proved the statement by contraposition.

- Ron attempted a proof by contradiction: Assume $\frac{x}{y}$ is irrational. Further suppose both x and y are rational in order to get a contradiction. By the definition of the rational numbers $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers $a, b \neq 0, c, d \neq 0$. Thus, the ratio $\frac{x}{y} = \frac{ad}{bc}$, where ad and $bc \neq 0$ are integers, is also rational by definition. This contradicts the irrationality of $\frac{x}{y}$. Therefore, either x or y has to be irrational.

- Hermione attempted a proof by contraposition: Assume both x and y are rational numbers. Then by the definition of the rational numbers $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers $a, b \neq 0, c, d \neq 0$. Thus, the ratio $\frac{x}{y} = \frac{ad}{bc}$, where ad and $bc \neq 0$ are integers, is also rational by definition. Thus, we have proved the statement by contraposition.

- Harry attempted a proof by contradiction: Assume $\frac{x}{y}$ is rational and x is irrational or y is irrational.

  Case 1: Both x and y are irrational. Then, the ratio $\frac{x}{y}$ is also irrational which contradicts the rationality of $\frac{x}{y}$.

  Case 2: y is rational and x is irrational. Since $x = \frac{x}{y} \cdot y$, x is a product of two rational numbers $\frac{x}{y}$ and y, which must be a rational number. This contradicts the irrationality of x.

  Case 3: x is rational and y is irrational. Since $\frac{x}{y}$ is rational, its reciprocal $\frac{y}{x}$ is also rational. Furthermore, $y = \frac{y}{x} \cdot x$. That is, y is a product of two rational numbers $\frac{y}{x}$ and x, which is rational. This contradicts the irrationality of y.

In all three possible cases, we obtain a contradiction. Thus, we have proved the statement using proof by contradiction.

7. Consider the following two proofs of the theorem below. Which one is a more appropriate choice and why?

   For all real numbers x, if x^2 is irrational then x is irrational.

   (a) Proof by contraposition: Assume x is an arbitrary rational number. By definition $x = \frac{a}{b}$ for some integers $a, b \neq 0$. Then $x^2 = \frac{a^2}{b^2}$. Since a^2 and $b^2 \neq 0$ are both integers, by definition x^2 is a rational number.

   (b) Proof by contradiction: Assume there is a real number x such that x^2 is irrational but x is rational. By definition $x = \frac{a}{b}$ for some integers $a, b \neq 0$. Then $x^2 = \frac{a^2}{b^2}$. Since a^2 and $b^2 \neq 0$ are both integers, by definition x^2 is a rational number which contradicts the irrationality of x^2.

8. Give a short proof, maximum 4 to 6 sentences, for the following statements:

   (a) For all real numbers x, $x^2 \geq 0$.

   (b) There is an integer x such that $2x + 3 = 5$.

   (c) For every rational number x, $x \neq 2$, there is a rational number y such that $xy - 2y = 4$.

   (d) There is a real number $x \neq 0$ such that $\cos(y) = \cos(x + y)$ for all real numbers y.

   (e) For all integers n there is an odd integer k such that $n \leq k + 2 < n + 2$.

   (f) There does not exist a positive integer n such that $2n + 5 = 6$.

   (g) For all real numbers x, $(x^2 < 0 \rightarrow x + 1 = 5)$.

9. Disprove (i.e., show that the statement is false) the following statements:

   (a) For all real numbers x, $3x + 2 > 5$.

   (b) There is a real number x such that $x^2 = -1$.

   (c) There exist positive integers x and y such that $x + y = 1$.

   (d) For all integers x and y, $x + y = 1$.

   (e) For all real numbers x, $(x^2 \leq 0 \rightarrow x + 1 = 5)$.

10. Which proofs are correct and which ones are incorrect? If you think a proof is incorrect explain why.

    (a) *There exists an integer n such that $3n + 4 = 10$.*

       i. Proof: If $3n + 4 = 10$, then $3n = 6$. This implies $n = 2$. Therefore, there exists an integer n such that $3n + 4 = 10$.

       ii. Proof: Assume n is an arbitrary integer such that $n = 2$. Then $3n + 4 = 10$.

iii. Proof: Let $n = 2$. Then $3 \cdot 2 + 4 = 10$.

iv. Proof: Let $n = 2$. Then $3n + 4 = 3 \cdot 2 + 4 = 10$.

(b) *For all positive integers n, there is an integer k such that $n < k < n + 2$.*

i. Proof: Assume n is an arbitrary positive integer. Let $k = n + 1$. Then $n < n + 1 < n + 2$.

ii. Proof: Assume n is an arbitrary positive integer such that $k = n + 1$. Then $n < k < n + 2$.

iii. Proof: Assume n and k are arbitrary positive integers such that $k = n + 1$. Then $n < k < n + 2$.

iv. Proof: Assume n is an arbitrary positive integer. Let k be a positive integer satisfying the inequality $n < k < n + 2$. Therefore we have proved that for all positive integers n, there is an integer k such that $n < k < n + 2$.

v. Proof: Assume n in an arbitrary positive integer. Let $k = n + 1$. Then $n < k = n + 1 < n + 2$.

vi. Proof: Assume n arbitrary positive. Let $k = n + 1$. Then $n < k < n + 2 = n < n + 1 < n + 2$.

(c) *There exists an integer k such that $k + n = n + 2$ for all integers n.*

i. Assume k and n are arbitrary integers. Pick $k = 2$. Then $k + n = n + 2$.

ii. Let $k = 2$. Then $k + n = n + 2$ for all integers n.

iii. Assume $k + n = n + 2$ where k, n are integers. Subtracting n from both sides, we obtain $k = 2$. Thus, we have proved that $k = 2$ satisfies the equation $k + n = n + 2$ for all integers n.

iv. Let $k = 2$ and $n = 1$. Then $k + n = n + 2$. Therefore, if $k = 2$, then we can see that $k + n = n + 2$ for all integers n.

11. Select the correct options for the following statements? Explain your reasoning.

(a) Let $a = \pi \cdot b$ such that b is rational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

(b) Let $a = \pi \cdot b$ such that b is irrational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

(c) Let $a = \pi + b$ such that b is rational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

(d) Let $a = \pi + b$ such that b is irrational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

12. Use a direct proof which may include two or more cases to justify:

    For all integers n, $n^3 + 3n + 4$ is even.

13. Use a direct proof which may include two or more cases to justify:

    For all integers n, $1 + (-1)^n(2n-1)$ is an integer multiple of 4.

14. Prove by contraposition:

    For any real number x, if $x^2 + 5x < 0$, then $x < 0$.

15. Prove by contradiction:

    There are no integers a and b such that $18a + 6b = 1$.

16. Prove the following statement:

    If $a^2 + b^2 = c^2$ for some integers a, b, c , then a is even or b is even.

Solutions:

1. Complete the definitions of the following concepts:

   (a) A mathematical proof is a valid argument which verifies that a mathematical statement is true.

   (b) An axiom is a true mathematical statement that is self-evident. It does not need a proof. Each area of mathematics is defined by a set of axioms.

   (c) A theorem is an important true mathematical statement that has been proved.

   (d) A proposition is a true mathematical statement which less important than a theorem.

   (e) A lemma is a mathematical result, usually of a technical nature, that is used in the proof of a theorem.

   (f) A corollary is a mathematical result that is a consequence of a theorem, usually emphasizing a particular case of a theorem.

2. Prove the following statement using direct proof techniques:

   For all integers n, m and p, if $n+p$ is odd and $m+p$ is odd, then $n+m$ is even.

   Proof. Let m, n and p be integers. Assume that $n+p$ and $m+p$ are odd. Then by definition of odd numbers, there exist k and ℓ integers so that

   $m+p = 2k+1$, and

   $n+p = 2\ell+1$.

   Then, using the above equations and basic algebra,

   $$m+n = (m+p)+(n+p)-2p = (2k+1)+(2\ell+1)-2p = 2(k+\ell-p+1). \qquad (1)$$

   Since $k+\ell-p+1$ is an integer, (1) shows that $m+n$ is even. ■

3. Prove the following statements using contraposition:

   For all real numbers r and s, if $r+s$ is irrational, then $2r-s$ is irrational or $2s-r$ is irrational.

   Proof. Let r and s be real numbers. Assume that $2r-s$ is rational and $2s-r$ is rational. By the definition of rational numbers, there exist integers m, n, k, and ℓ, $n \neq 0$ and $\ell \neq 0$ so that

   $2r - s = \dfrac{m}{n}$, and

   $2s - r = \dfrac{k}{\ell}$.

   Using the above equations and basic algebra, we write

   $$r+s = (2r-s)+(2s-r) = \frac{m}{n}+\frac{k}{\ell} = \frac{m\ell+nk}{n\ell}.$$

   Since $m\ell+nk$ and $n\ell$ are integers and $n\ell \neq 0$, by the definition of rational numbers, $r+s$ is rational. ■

4. Prove the following statements by contradiction:

   For all odd integers a, b, and c, if x is a solution of the equation $ax^2+bx+c=0$, then x is irrational.

Proof. Let a, b, and c be odd integers and assume that x is a solution to the equation $ax^2 + bx + c = 0$. To get a contradiction assume that x is rational.

By definition of rational numbers, there exist m and n integers, $n \neq 0$, so that

$$x = \frac{m}{n}.$$

Without loss of generality, we can assume that m and n are relatively prime, in particular, at least one of them is odd.

Since x is a solution to the given quadratic equation, we infer that

$$a\left(\frac{m}{n}\right)^2 + b\frac{m}{n} + c = 0$$

Multiplying the equation by n^2, we get

$$am^2 + bmn + cn^2 = 0. \qquad (1)$$

Recall that a, b, and c are odd. We need to consider the following cases:

For the case when m is odd and n is even, equation (1) reads "odd + even + even =0,"

For the case when n is odd and m is even, equation (1) reads "even + even + odd =0," and

For the case when both m and n are odd, equation (1) reads "odd + odd + odd =0."

In all of the cases above, we obtain a contradiction, since 0 is an even number and the left hand side of each equation is odd. ■

5. Find the correct proof for the following theorem:

The sum of an odd integer x and an even integer y is odd.

(a) Assume x is an arbitrary even integer and y is an arbitrary odd integer. By definition $x = 2n$ and $y = 2n + 1$ for some integer n. Then $x + y = 2n + (2n + 1) = 4n + 1 = 2 \cdot (2n) + 1$ which is odd by the definition of odd integers.

Incorrect. We cannot use the same variable notation n for both x and y (as even and respectively odd) since they are arbitrary integers and not necessarily consecutive.

(b) Let $x = 4$ and $y = 3$. Then $x + y = 7$. Since $7 = 2 \cdot 3 + 1$, by definition 7 is odd. Therefore the sum of an even and odd integer is odd.

Incorrect. To prove that a property is true for arbitrary values of one or more variables, we cannot simply "prove by example."

(c) Assume x is an arbitrary even and y is an arbitrary odd integer. By definition $x = 2n$ and $y = 2m + 1$ for some integers n, m. Then $x + y = 2n + (2m + 1) = 2(n + m) + 1$. Since $n + m$ is an integer, $x + y$ is odd by the definition of odd integers.

Correct.

(d) Assume x is an arbitrary even and y is an arbitrary odd integer. By definition $x = 2n$ and $y = 2m + 1$ for arbitrary integers n, m. Then $x + y = 2n + (2m + 1) = 2(n + m) + 1$. Since $n + m$ is an integer, $x + y$ is odd by the definition of odd integers.

Incorrect. n and m are not arbitrary, they are determined by the values of x and y respectively.

(e) We use proof by contraposition. Assume $x + y$ is even. By the definition of even integers, $x + y = 2n$ for some integer n. Then
Case 1: $x = 2n - 2$ and $y = 2$.

Case 2: $x = 2n - 1$ and $y = 1$.

In both cases x and y are either both even or both odd integers. Thus, the theorem is proved by contraposition.

Incorrect. There are many more configurations for the values x and y.

(f) We use proof by contradiction to prove this theorem. Assume x is an arbitrary even, y is an arbitrary odd integer and $x + y$ is even. Then $-y$ is also an odd integer. Since $x = (x + y) + (-y)$, x is a sum of an even and an odd integer, x is odd. This contradicts the evenness of x.

Incorrect. It is a circular argument. The proof is using the theorem that "the sum of an even and an odd integer is odd" which is what we are trying to prove here. Basically, this argument has the following form: A is true because A is true.

6. Several students at Hogwarts took the OWL exam. The students had to prove the following theorem from definitions and principles. Below you can see Ron's, Harry's, Ginny's and Hermione's attempts. Which student(s) proved the statement correctly?

For all real numbers x and $y \neq 0$, if $\frac{x}{y}$ is irrational, then x is irrational or y is irrational.

- Ginny attempted a proof by contraposition: Assume $\frac{x}{y}$ is rational. By the definition of rational $\frac{x}{y} = \frac{p}{q}$ for some integers $p, q \neq 0$. Then $x = p$ and $y = q$. Since p and q are integers, x and y are rational numbers. Thus, we have proved the statement by contraposition.

  Incorrect. Ginny is trying to prove the inverse statement and not the contrapositive. Furthermore, $\frac{x}{y} = \frac{p}{q}$ does not imply that $x = p$ and $y = q$. For example, if $x = \ell \cdot p$ and $y = \ell \cdot q$ for some non-zero real number ℓ, the quotient is still $\frac{p}{q}$.

- Ron attempted a proof by contradiction: Assume $\frac{x}{y}$ is irrational. Further suppose both x and y are rational in order to get a contradiction. By the definition of the rational numbers $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers $a, b \neq 0, c, d \neq 0$. Thus, the ratio $\frac{x}{y} = \frac{ad}{bc}$, where ad and $bc \neq 0$ are integers, is also rational by definition. This contradicts the irrationality of $\frac{x}{y}$. Therefore, either x or y has to be irrational.

  Technically correct. However, it is unnecessary to use proof by contradiction in this problem. Notice that a proof by contraposition (indicated by blue color in Ron's proof) is wrapped inside a proof by contradiction. We should not use proof by contradiction when a simpler proof is available. See Hermione's proof below.

- Hermione attempted a proof by contraposition: Assume both x and y are rational numbers. Then by the definition of the rational numbers $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some integers $a, b \neq 0, c, d \neq 0$. Thus, the ratio $\frac{x}{y} = \frac{ad}{bc}$, where ad and $bc \neq 0$ are integers, is also rational by definition. Thus, we have proved the statement by contraposition.

  Correct.

- Harry attempted a proof by contradiction: Assume $\frac{x}{y}$ is rational and x is irrational or y is irrational.

  Case 1: Both x and y are irrational. Then the ratio $\frac{x}{y}$ is also irrational which contradicts the rationality of $\frac{x}{y}$.

  Case 2. y is rational and x is irrational. Since $x = \frac{x}{y} \cdot y$, x is a product of two rational numbers $\frac{x}{y}$ and y, which must be a rational number. This contradicts the irrationality of x.

Case 3: x is rational and y is irrational. Since $\frac{x}{y}$ is rational, its reciprocal $\frac{y}{x}$ is also rational. Furthermore, $y = \frac{y}{x} \cdot x$. That is, y is a product of two rational numbers $\frac{y}{x}$ and x, which is rational. This contradicts the irrationality of y.

In all three possible cases, we obtain a contradiction. Thus, we have proved the statement using proof by contradiction.

Harry negates the statement incorrectly. The negation of "if $\frac{x}{y}$ is irrational, then x is irrational or y is irrational" is "$\frac{x}{y}$ is irrational and x is rational and y is rational." Furthermore, in Case 1 the ratio of two irrational numbers is not necessarily irrational. For example, $\frac{3\pi}{\pi} = 3$, although both the numerator and the denominator are irrationals but the ratio is an integer.

7. Consider the following two proofs of the theorem below. Which one is a more appropriate choice and why?

   For all real numbers x, if x^2 is irrational, then x is irrational.

   (a) Proof by contraposition: Assume x is an arbitrary rational number. By definition $x = \frac{a}{b}$ for some integers $a, b \neq 0$. Then $x^2 = \frac{a^2}{b^2}$. Since a^2 and $b^2 \neq 0$ are both integers, by definition x^2 is a rational number.

   (b) Proof by contradiction: Assume there is a real number x such that x^2 is irrational but x is rational. By definition $x = \frac{a}{b}$ for some integers $a, b \neq 0$. Then $x^2 = \frac{a^2}{b^2}$. Since a^2 and $b^2 \neq 0$ are both integers, by definition x^2 is a rational number which contradicts the irrationality of x^2.

   Proof (a) is the better proof. While the second proof is technically correct, it is just a proof by contraposition wrapped inside a proof by contradiction, just like Ron's proof in the previous example.

8. Give a short proof using maximum 4 to 6 sentences for the following statements:

   (a) For all real numbers x, $x^2 \geq 0$.

   Proof. Let x be a real number. Consider the following cases:
   if x is positive, $x^2 = x \cdot x$ is "positive $\cdot$ positive $=$ positive," so $x^2 > 0$,
   if x is negative, $x^2 = x \cdot x$ is "negative $\cdot$ negative $=$ positive," so $x^2 > 0$, and
   if $x = 0$, then $x^2 = 0$.
   In all cases, $x^2 \geq 0$. ■

   (b) There is an integer x such that $2x + 3 = 5$.

   Proof. Let $x = 1$. Then $2x + 3 = 2 \cdot 1 + 3 = 5$. ■

   (c) For every rational number x, $x \neq 2$, there is a rational number y such that $xy - 2y = 4$.

   Proof. Let x be a rational number, $x \neq 2$. Define $y = \dfrac{4}{x-2}$. Then y is well-defined and is rational, as the quotient of two rational numbers. Also,
   $$xy - 2y = x\frac{4}{x-2} - 2\frac{4}{x-2} = \frac{4x-8}{x-2} = \frac{4(x-2)}{x-2} = 4,$$
   hence y is the value whose existence the theorem asserts. ■

   (d) There is a real number $x \neq 0$ such that $\cos(y) = \cos(x + y)$ for all real numbers y.

   Proof. Let $x = 2\pi$. Then $\cos(y) = \cos(x+y)$ for all real numbers y, since the cosine function is periodic with period 2π, i.e., $\cos(y) = \cos(2\pi + y)$. ■

(e) For all integers n there is an odd integer k such that $n \leq k+2 < n+2$.

Proof. Let n be an arbitrary integer.
If n is odd, let $k = n-2$, which is also odd, and $n = (n-2)+2 = k+2 < n+2$.
If n is even, let $k = n-1$, which is odd, and $n \leq n+1 = (n-1)+2 = k+2 < n+2$. ■

(f) There does not exist a positive integer n such that $2n+5 = 6$.

Proof 1. For any integer n, $2n+5$ is odd and 6 is even, hence the equality cannot happen. ■

Proof 2. Since $n \geq 1$, $2n+5 \geq 7$. Thus, $2n+5 \neq 6$. ■

(g) For all real numbers x, $(x^2 < 0 \to x+1 = 5)$.

Proof. For each real number x the premise of the conditional statement is false, thus the conditional statement is true for all real numbers x. ■

9. Disprove (i.e., show that the statement is false) the following statements:

(a) For all real numbers x, $3x+2 > 5$.

We disprove the statement by giving a counter example. Let $x = -2$. Then $3x+2 = -4$ and $-4 \not> 5$.

(b) There is a real number x such that $x^2 = -1$.

We will show that there is no real number x such that $x^2 = -1$. Since for any real number x, $x^2 \geq 0$, and hence there is no value of x for which x^2 can be equal to -1, which is negative.

(c) There exist positive integers x and y such that $x+y = 1$.

We will show that there are no positive integers x and y such that $x+y = 1$.
If x and y are (arbitrary) positive integers,
$$x \geq 1 \quad \text{and } y \geq 1,$$
so $x+y \geq 2$. Hence $x+y$ cannot be equal to 1 for any positive integers x and y.

(d) For all integers x and y, $x+y = 1$.

We use a counter example to disprove the statement.
Let $x = 3$ and $y = -5$, then $x+y = -2 \neq 1$.

(e) For all real numbers x, $(x^2 \leq 0 \to x+1 = 5)$.

We give a counter example to show that $\forall x(x^2 \leq 0 \to x+1 = 5)$ is false. For $x = 0$, the premise is true and the conclusion is false, thus the conditional statement $(x^2 \leq 0 \to x+1 = 5)$ is false. ■

10. Check whether the following proofs are correct or incorrect. It you think a proof is incorrect explain why.

(a) *There exists an integer n such that $3n+4 = 10$.*

i. Proof: If $3n+4=10$, then $3n=6$. This implies $n=2$. Therefore, there there exists an integer n such that $3n+4=10$.

Incorrect. This argument only shows that if a solution n exists for the equation $3n+4=10$, then $n=2$. Assuming existence does not prove an existence.

ii. Proof: Assume n is an arbitrary integer such that $n=2$. Then $3n+4=10$.

Incorrect. If you assume n is arbitrary, then you cannot set its value at 2. Then n is not arbitrary anymore.

iii. Proof: Let $n=2$. Then $3\cdot 2+4=10$.

Incorrect. The conclusion does not refer to the variable n. $3\cdot 2+4=10$ is always true regardless of the value n. One cannot make a variable definition and then draw a conclusion from it that makes no reference to that variable.

iv. Proof: Let $n=2$. Then $3n+4=3\cdot 2+4=10$.

Correct.

(b) *For all positive integers n, there is an integer k such that $n<k<n+2$.*

i. Proof: Assume n is an arbitrary positive integer. Let $k=n+1$. Then $n<n+1<n+2$.

Incorrect. The conclusion must refer to the variable k. $n<n+1<n+2$ is always true regardless of the value of k. One cannot make a variable definition and then draw a conclusion from it that makes no reference to that variable.

ii. Proof: Assume n is an arbitrary positive integer such that $k=n+1$. Then $n<k<n+2$.

Incorrect. One cannot use the "such that" phrase to make a definition. To define k as $n+1$, we say "let $k=n+1$", "pick $k=n+1$" or "select $k=n+1$."

iii. Proof: Assume n and k are arbitrary positive integers such that $k=n+1$. Then $n<k<n+2$.

Incorrect. There are two conceptual mistakes in this proof.
The proof above assumes that k is an arbitrary integer and then immediately contradicts itself by stating that $k=n+1$.
We should not use "such that" phrase to make a definition. See the comment above.

iv. Proof: Assume n is an arbitrary positive integer. Let k be a positive integer satisfying the inequality $n<k<n+2$. Therefore we have proved that for all positive integers n, there is an integer k such that $n<k<n+2$.

Incorrect. It is fallacy of affirming the conclusion.

v. Proof: Assume n is an arbitrary positive integer. Let $k=n+1$. Then $n<k=n+1<n+2$.

Correct.

vi. Proof: Assume n is an arbitrary positive integer. Let $k=n+1$. Then $n<k<n+2=n<n+1<n+2$.

Incorrect. Don't put equal sign between equivalent inequalities.

The notation "$n < k < n+2 = n < n+1 < n+2$" means that "$n < k$ and $k < n+2$ and $n+2 = n$ and $n < n+1$ and $n+1 < n+2$." Of course, $n = n+2$ is false.

The notation "$n < k < n+2 = n < n+1 < n+2$" does NOT mean that the inequality $n < k < n+2$ becomes $n < n+1 < n+2$ for $k = n+1$.

The correct way to write the conclusion is $n < k = n+1 < n+2$.

(c) *There exists an integer k such that $k+n = n+2$ for all integers n.*

i. Assume k and n are arbitrary integers. Pick $k = 2$. Then $k+n = n+2$.

Incorrect. k is not arbitrary, k must be a specific integer for which $k+n = n+2$ for all integers n.

ii. Let $k = 2$. Then $k+n = n+2$ for all integers n.

Correct.

iii. Assume $k+n = n+2$ where k, n are integers. Subtracting n from both sides, we obtain $k = 2$. Thus, we have proved that $k = 2$ satisfies the equation $k+n = n+2$ for all integers n.

Incorrect. This argument assumes the conclusion and only shows that, if there exists a k for which $k+n = n+2$ for all integers n, then $k = 2$. It was never shown that $k = 2$ solves the equation $k+2 = n+2$ for all n. To prove existence, one cannot start the argument by assuming the existence.

iv. Let $k = 2$ and $n = 1$. Then $k+n = n+2$. Therefore, if $k = 2$, then we can see that $k+n = n+2$ for all integers n.

Incorrect. We have to show that $k = 2$ satisfies $k+n = n+2$ for all n, and not just for $n = 1$.

11. Select the correct option for the following statements. Explain your reasoning.

(a) Let $a = \pi \cdot b$ such that b is rational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

If $b = 0$, then $a = 0$, which is rational. If $b = 1$,then $a = \pi$, which is irrational.
In general, if $b \neq 0$ and rational, then a will be irrational.

(b) Let $a = \pi \cdot b$ such that b is irrational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

If $b = \pi$, then $a = \pi^2$, which is is irrational. If $b = \dfrac{3}{\pi}$, then $a = 3$ is rational.

(c) Let $a = \pi + b$ such that b is rational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

If a were rational, then $a - b = \pi$ would be rational, which is not possible, since π is irrational.

(d) Let $a = \pi + b$ such that b is irrational. Then

i. a must be rational.

ii. a must be irrational.

iii. a could be rational or irrational.

If $b = \pi$, then $a = 2\pi$ is irrational, if $b = -\pi$, then $a = 0$ is rational.

12. Use a direct proof which may include two or more cases to justify:

For all integers n, $n^3 + 3n + 4$ is even.

Proof. Suppose n is an arbitrary integer.
If n is odd, then n^3 is odd, and $3n$ is odd, hence $n^3 + 3n + 4$ is odd + odd + even, so it is even.
If n is even, then n^3 is even and $3n$ is even, so $n^3 + 3n + 4$ is even + even + even, so it is even.
Hence in any case, $n^3 + 3n + 4$ is even. ■

13. Use a direct proof which may include two or more cases to justify:

For all integers n, $1 + (-1)^n(2n - 1)$ is a multiple of 4.

Proof. Suppose n is an arbitrary integer.
In case n is even, then there exists an integer k so that $n = 2k$. Hence,
$$1 + (-1)^n(2n - 1) = 1 + (-1)^{2k}(2(2k) - 1) = 1 + (4k - 1) = 4k.$$
In case n is odd, then there exists an integer k so that $n = 2k + 1$. Hence,
$$1 + (-1)^n(2n - 1) = 1 + (-1)^{2k+1}(2(2k + 1) - 1) = 1 - (4k + 2 - 1) = -4k.$$
In either case, $1 + (-1)^n(2n - 1)$ is an integer multiple of 4. ■

14. Prove by contraposition:

For any real number x, if $x^2 + 5x < 0$, then $x < 0$.

Proof. Suppose x is a real number. Assume $x \geq 0$. Then $x + 5 \geq 5$ and
$$x^2 + 5x = x(x + 5) \geq 0$$
as both factors are nonnegative. ■

15. Prove by contradiction:

    There are no integers a and b such that $18a + 6b = 1$.

    Proof. To get a contradiction assume that there exists two integers a and b such that

    $18a + 6b = 1. \qquad (1)$

    But $18a + 6b = 2(9a + 3b)$ is even (since $9a + 3b$ is an integer) and 1 is odd. Hence (1) is a contradiction. ■

16. Prove the following statement:

    If $a^2 + b^2 = c^2$ for some integers a, b, c, then a is even or b is even.

    Proof by contradiction. Let a, b and c be integers such that

    $a^2 + b^2 = c^2. \qquad (1)$

    Without loss of generality, we can assume that a, b, and c do not have a common positive factor larger than 1. (In the opposite case, we can just divide the equation by the largest such factor.)

    To get a contradiction, assume by that a and b are both odd. Then

    $a^2 + b^2$ is even.

    Hence,

    c^2 is even, and, consequently, c is even.

    Representing even and odd integers by their definition, there exist integers k, ℓ, j such that

    $a = 2k + 1$, $b = 2\ell + 1$ and $c = 2j$.

    Substituting these into (1), we get:

    $(2k + 1)^2 + (2\ell + 1)^2 = (2j)^2$

    or equivalently

    $4(k^2 + k + \ell^2 + l) + 2 = 4j^2.$

    Dividing by 2,

    $2(k^2 + k + \ell^2 + l) + 1 = 2j^2,$

    which is a contradiction, since an odd number (left hand side) cannot be equal to an even number (right hand side). ■