Concepts:

- Define the concepts of tautology, contradiction, contingency, and logical equivalence.
- Be familiar with the basic laws of logical equivalence.
- Use laws of logical equivalence to simplify compound propositions and identify them as tautologies, contradictions or contingencies or to prove logical equivalences.

Problems:

1. Let p and q be propositional variables. Identify which of the following are tautologies. Recall that $\oplus$ denotes the "exclusive or" operator. Show your reasoning.

   (a) $p \to p$

   (b) $p \wedge p$

   (c) $p \oplus p$

   (d) $p \leftrightarrow p$

   (e) $p \vee p$

   (f) $\neg p \vee (q \vee p)$

   (g) $p \vee (\neg p \wedge q)$

   (h) $(p \to q) \vee p$

   (i) $(q \to p) \vee p$

2. Let p and q be propositional variables. Use logical equivalences to fully simplify $(p \to q) \to q$ until you have only one logical operator present in the new equivalent expression.

   (a) $p \vee q$

   (b) $\neg p \wedge q$

   (c) $p \vee \neg q$

   (d) $p \wedge q$

   (e) None of these.

3. Let a, b, and c be propositional variables. Which of the following is the resultant expression when we apply the distributive law to $(a \wedge b) \vee c$:

   (a) $(a \wedge c) \vee (b \wedge c)$

(b) $(a \vee c) \wedge (b \vee c)$

(c) $(a \wedge c) \wedge (b \wedge c)$

(d) $(a \vee c) \vee (b \vee c)$

4. Which of the following expressions is/are equivalent to the conditional statement $p \to q$.

(a) If p, then q.

(b) p only if q.

(c) q only if p.

(d) p is a necessary condition for q.

(e) q when p.

(f) q is a necessary condition for p.

(g) q if p.

(h) p is a sufficient condition for q.

(i) q is a sufficient condition for p.

(j) $\neg p \vee q$

(k) $\neg(p \wedge \neg q)$

(l) $\neg(p \vee \neg q)$

(m) $q \to p$

(n) $p \wedge \neg q$

5. Check whether the following statements are true or false. Show reasoning when they are true.

(a) Conjunction can be expressed (in logical equivalence sense) using disjunction and negation.

(b) Conjunction can be expressed (in logical equivalence sense) using conditional and negation.

(c) Disjunction can be expressed (in logical equivalence sense) using conjunction and negation.

(d) Disjunction can be expressed (in logical equivalence sense) using conditional and negation.

(e) Conditional can be expressed (in logical equivalence sense) using disjunction and negation.

(f) Conditional can be expressed (in logical equivalence sense) using conjunction and negation.

6. Use logical equivalences to verify that the expression $(p \wedge \neg q) \vee (q \vee \neg p)$ is a tautology. Name the logical identities in the order they are used in the simplification process. Use at most 8 steps in your solution.

7. Use logical equivalences to verify that $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$. Name the logical identities in the order they are used. Use at most 8 steps in your solution.

8. Select which of the following statements is/are logically equivalent to $p \to (p \to q)$:

   (a) $\neg p \wedge q$

   (b) $p \vee \neg q$

   (c) $\neg p \vee \neg q$

   (d) $p \to q$

   (e) $q \to p$

9. Consider the statement *"If you smoke, then you will get sick."* Select which of the following statements is/are NOT logically equivalent to the conditional above.

   (a) You will get sick only if you smoke.

   (b) Smoking is a sufficient condition for getting sick.

   (c) Being sick is a necessary condition for smoking.

   (d) If you are not sick, then you did not smoke.

10. Which of the following logical expressions is/are logically equivalent to the negation of the conditional statement $p \to q$:

   (a) $\neg p \to q$

   (b) $p \to \neg q$

   (c) $\neg p \to \neg q$

   (d) $\neg q \to p$

   (e) $q \to \neg p$

   (f) $\neg q \to \neg p$

   (g) None of these.

11. Select which of the following four options is correct considering the two statements;

    1. $(p \wedge q) \vee (p \wedge \neg q)$,

    2. $(p \vee q) \wedge (p \vee \neg q)$.

    (a) 1 is a tautology and 2 is a contradiction.

    (b) 1 is a tautology, 2 is a contingency.

    (c) 1 is a contingency, 2 is a contradiction.

    (d) Both statements are contingencies.

12. Check whether the following statements are true or false. Show your reasoning.

    (a) A conditional is equivalent to its converse.

    (b) A conditional is equivalent to its contrapositive.

    (c) A conditional is equivalent to its inverse.

    (d) Converse and inverse of a conditional are equivalent.

    (e) Converse and contrapositive of a conditional are equivalent.

    (f) Inverse and contrapositive of a conditional are equivalent.

13. Which of the following statements is/are logically equivalent to

    "If the sky is not cloudy, then the Sun is shining."

    (a) The Sun is not shining unless the sky is not cloudy.
    (b) The Sun is shining only if the sky is not cloudy.
    (c) Sunny weather is a sufficient condition for not having clouds on the sky.
    (d) Cloudy weather is a necessary condition for the Sun to shine.
    (e) Sunny weather is a necessary condition for the sky not to be cloudy.

    Note: *"Sunny weather"* is equivalent to *"Sun is shining."*

Solutions:

1. Let p and q be propositional variables. Identify which of the following are tautologies. Recall that $\oplus$ denotes the "exclusive or" operator. Show your reasoning.

   (a) $p \to p$

   (b) $p \wedge p$ The expression is False when p is False.

   (c) $p \oplus p$ The expression is False when p is True or p is False. So the expression is in fact a contradiction.

   (d) $p \leftrightarrow p$

   (e) $p \vee p$ The expression is False when p is False.

   (f) $\neg p \vee (q \vee p)$

   (g) $p \vee (\neg p \wedge q)$ The expression is False when p and q are False.

   (h) $(p \to q) \vee p$

   (i) $(q \to p) \vee p$ The expression is False when p is False, and q is True.

2. Let p and q be propositional variables. Use logical equivalences to fully simplify $(p \to q) \to q$ until you have only one logical operator present in the new equivalent expression.

   (a) $p \vee q$

$$
\begin{aligned}
(p \to q) \to q &\equiv \neg(\neg p \vee q) \vee q && \text{(Definition of Conditional twice)}\\
&\equiv (\neg(\neg p) \wedge \neg q) \vee q && \text{(De Morgan Law)}\\
&\equiv (p \wedge \neg q) \vee q && \text{(Double Negation)}\\
&\equiv q \vee (p \wedge \neg q) && \text{(Commutative Law)}\\
&\equiv (q \vee p) \wedge (q \vee \neg q) && \text{(Distributive Law)}\\
&\equiv (q \vee p) \wedge T && \text{(Negation Law)}\\
&\equiv q \vee p && \text{(Identity Law)}\\
&\equiv p \vee q && \text{(Commutative Law)}
\end{aligned}
$$

   (b) $\neg p \wedge q$

   (c) $p \vee \neg q$

   (d) $p \wedge q$

   (e) None of these.

3. Let a, b, and c be propositional variables. Which of the following is the resultant expression when we apply the distributive law to $(a \wedge b) \vee c$:

   (a) $(a \wedge c) \vee (b \wedge c)$

   (b) $(a \vee c) \wedge (b \vee c)$ Correct.

   (c) $(a \wedge c) \wedge (b \wedge c)$

   (d) $(a \vee c) \vee (b \vee c)$

4. Which of the following expressions is/are equivalent to the conditional statement $p \to q$.

   (a) If p, then q. (equivalent)

   (b) p only if q. (equivalent)

   (c) q only if p.

   (d) p is a necessary condition for q.

   (e) q when p. (equivalent)

   (f) q is a necessary condition for p. (equivalent)

   (g) q if p. (equivalent)

   (h) p is a sufficient condition for q. (equivalent)

   (i) q is a sufficient condition for p.

   (j) $\neg p \vee q$. (equivalent)

   (k) $\neg(p \wedge \neg q)$. (equivalent)

   (l) $\neg(p \vee \neg q)$

   (m) $q \to p$

   (n) $p \wedge \neg q$

5. Check whether the following statements are true or false. Show reasoning when they are true.

   (a) Conjunction can be expressed using disjunction and negation.

   True: $a \wedge b \equiv \neg(\neg a \vee \neg b)$

(b) Conjunction can be expressed using conditional and negation.

True: $a \wedge b \equiv \neg(a \rightarrow \neg b)$

(c) Disjunction can be expressed using conjunction and negation.

True: $a \vee b \equiv \neg(\neg a \wedge \neg b)$

(d) Disjunction can be expressed using conditional and negation.

True: $a \vee b \equiv \neg a \rightarrow b$

(e) Conditional can be expressed using disjunction and negation.

True: $a \rightarrow b \equiv \neg a \vee b$

(f) Conditional can be expressed using conjunction and negation.

True: $a \rightarrow b \equiv \neg(a \wedge \neg b)$

6. Use logical equivalences to verify that the expression $(p \wedge \neg q) \vee (q \vee \neg p)$ is a tautology. Name the logical identities in the order they are used in the simplification process. Use at most 8 steps in your solution.

$$\begin{aligned}
(p \wedge \neg q) \vee (q \vee \neg p) &\equiv ((p \wedge \neg q) \vee q) \vee \neg p && \text{(Associativity of Disjunction)}\\
&\equiv (q \vee (p \wedge \neg q)) \vee \neg p && \text{(Commutative Law)}\\
&\equiv ((q \vee p) \wedge (q \vee \neg q)) \vee \neg p && \text{(Distributive Law)}\\
&\equiv ((q \vee p) \wedge T) \vee \neg p && \text{(Negation Law)}\\
&\equiv (q \vee p) \vee \neg p && \text{(Identity Law)}\\
&\equiv q \vee (p \vee \neg p) && \text{(Associative Law)}\\
&\equiv q \vee T && \text{(Negation Law)}\\
&\equiv T && \text{(Dominance Law)}
\end{aligned}$$

7. Use logical equivalences to verify that $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$. Name the logical identities in the order they are used in the process. Use at most 8 steps in your solution.

$$\begin{aligned}
(p \rightarrow r) \vee (q \rightarrow r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) && \text{(Definition of Conditional twice)}\\
&\equiv (\neg p \vee \neg q) \vee (r \vee r) && \text{(Commutative/Associative Laws for Disjunction multiple times)}\\
&\equiv \neg (p \wedge q) \vee r && \text{(De Morgan and Idempotent Laws)}\\
&\equiv (p \wedge q) \rightarrow r && \text{(Definition of Conditional)}
\end{aligned}$$

8. Select which of the following statements is/are logically equivalent to $p \rightarrow (p \rightarrow q)$:

(a) $\neg p \wedge q$

(b) $p \vee \neg q$

(c) $\neg p \vee \neg q$

(d) $p \to q$ Correct.

$$\begin{aligned} p \to (p \to q) &\equiv \neg p \vee (\neg p \vee q) && \text{(Definition of Conditional twice)} \\ &\equiv (\neg p \vee \neg p) \vee q && \text{(Associativity of Disjunction)} \\ &\equiv \neg p \vee q && \text{(Idempotent Law)} \\ &\equiv p \to q && \text{(Definition of Conditional)} \end{aligned}$$

(e) $q \to p$

9. Consider the statement *"If you smoke, then you will get sick."* Select which of the following statements is/are NOT logically equivalent to the conditional above.

   (a) You will get sick only if you smoke. (It is equivalent to the converse "if you get sick, then you smoked.")

   (b) Smoking it is a sufficient condition for getting sick.

   (c) Being sick is a necessary condition for smoking.

   (d) If you are not sick, then you did not smoke.

10. Which of the following logical expressions is/are logically equivalent to the negation of the conditional statement $p \to q$:

   (a) $\neg p \to q$

   (b) $p \to \neg q$

   (c) $\neg p \to \neg q$

   (d) $\neg q \to p$

   (e) $q \to \neg p$

   (f) $\neg q \to \neg p$

   (g) None of these.

   We construct the truth tables for all variations given above and observe that they have 3 rows true and 1 row false. However, the truth table for $\neg(p \to q)$ is false for 3 rows and true for 1 row. Therefore, the truth tables cannot agree.

   NOTE: The negation of $p \to q$ is logically equivalent to $p \wedge \neg q$. The negation of a conditional statement is NOT a conditional statement.

11. Select which of the following four options is correct considering the two statements;

   1. $(p \wedge q) \vee (p \wedge \neg q)$,

2. $(p \vee q) \wedge (p \vee \neg q)$.

(a) 1 is a tautology and 2 is a contradiction.

(b) 1 is a tautology and 2 is a contingency.

(c) 1 is a contingency and 2 is a contradiction.

(d) Both statements are contingencies. Correct.

$(p \wedge q) \vee (p \wedge \neg q) \equiv p \wedge (q \vee \neg q) \equiv p \wedge T \equiv p$ (by distributive property, negation and identity rules)
$(p \vee q) \wedge (p \vee \neg q) \equiv p \vee (q \wedge \neg q) \equiv p \vee F \equiv p$ (by distributive property, negation and identity rules)

12. Check whether the following statements are true or false. Show your reasoning.

(a) A conditional is equivalent to its converse.

False.

(b) A conditional is equivalent to its contrapositive.

True. This important equivalence $p \to q \equiv \neg q \to \neg p$ can be justified by truth table.

(c) A conditional is equivalent to its inverse.

False.

(d) Converse and inverse of a conditional are equivalent.

True. The inverse of a conditional statement is the contrapositive of the converse of the same statement.

(e) Converse and contrapositive of a conditional are equivalent.

False.

(f) Inverse and contrapositive of a conditional are equivalent.

False.

13. Check which statement is logically equivalent to

"If the sky is not cloudy, then the Sun is shining "

(a) The Sun is not shining unless the sky is not cloudy.
(b) The Sun is shining only if the sky is not cloudy.
(c) Sunny weather is a sufficient condition for not having clouds on the sky.
(d) Cloudy weather is a necessary condition for the Sun to shine.

(e) Sunny weather is a necessary condition for the sky not to be cloudy. Correct.

Define the notations; C = "It is cloudy" and S = "The sun is shining." The original statement can be expressed as $\neg C \to S$ and the proposed equivalent statements can be written as

(a) $C \to \neg S$
(b) $S \to \neg C$
(c) $S \to \neg C$
(d) $S \to C$
(e) $\neg C \to S$

Note: *"Sunny weather"* is equivalent to *"Sun is shining."*