Concepts:

- Define "relation" rigorously.
- Explain similarities and differences between the concepts of relations and functions. Explain how relations generalize the idea of functions.
- Be familiar with the different representations of relations: table, bubble diagram, or formal set, and be able to convert from one notation to another.
- Find the composition of two relations.
- Find the square of a given relation.
- Prove or disprove that a relation is reflexive, irreflexive, symmetric anti-symmetric, or transitive.
- Prove or disprove that a relation is an equivalence relation.

Problems:

1. Fill in the following definitions considering that A is a nonempty set:
   - A **relation on** A is defined as ...
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be **symmetric** if ...
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be **transitive** if ...
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be **reflexive** if ...
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be an **equivalence relation** if ...
   - If $\mathcal{R}_1$ and $\mathcal{R}_2$ are relations on A, then $\mathcal{R}_1 \circ \mathcal{R}_2$ represents ...

2. Check whether the following statements are true or false. Justify your answer briefly.
   (a) If a relation is symmetric, then it is not antisymmetric.
   (b) A relation is either reflexive or irreflexive.
   (c) If A is a nonempty set, then any relation on A represents the graph of a function $f : A \to A$.
   (d) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are reflexive on a set A then $\mathcal{R}_1 \circ \mathcal{R}_2$ is reflexive on A as well.
   (e) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are transitive on a set A, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is transitive on A as well.
   (f) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are transitive on a set A, then $\mathcal{R}_1 \cup \mathcal{R}_2$ is transitive on A as well.
   (g) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are reflexive on a set A, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive on A as well.

3. Determine if the relation is symmetric, antisymmetric, reflexive, irreflexive, or transitive on the set $A = \{1, 2, 3, 4\}$.

(a) $\mathcal{R}_1 = \{(1,1),(2,3),(2,4),(3,2),(3,4),(4,4)\}$.

(b) $\mathcal{R}_2 = \{(1,1),(1,2),(1,3),(2,1),(3,1),(4,4)\}$.

4. Given the following relations, check whether they are equivalence relations:

   (a) x is logically equivalent to y, on the set of all propositions.

   (b) $x \subseteq y$, on the set of all subsets of real numbers $\mathbb{R}$.

   (c) $f(x)$ is big-$\mathcal{O}$ of $g(x)$, on the set of real valued functions on the domain of real numbers.

   (d) $f(x)$ is big-Θ of $g(x)$, on the set of real valued functions on the domain of real numbers.

   (e) x divides y, on the set of non-zero integers.

   (f) $x \equiv y \pmod 7$, on the set of integers.

5. Given the following relations on the set of people, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

   (a) x is taller than y

   (b) x is born the same year as y

   (c) x and y have a common grandparent

6. Given the following relations on the set of integers, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

   (a) x and y have a common prime factor

   (b) $x - y$ is a multiple of 5

7. Given the following relations on the set of positive integers, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

   (a) $x - y = 1$

   (b) $|x - y| \geq 1$

8. Given the following relations on the set of real numbers, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

   (a) $x - y$ is a rational number

   (b) $x \cdot y \geq 0$

9. Given two relations on the set $\{1, 2, 3\}$ by the following tables:

$\mathcal{R}_1$:

	1	2	3
1	×		
2		×	×
3	×		

$\mathcal{R}_2$:

	1	2	3
1			×
2	×	×	
3			×

Construct the tables for the following.

(a) $\mathcal{R}_1 \circ \mathcal{R}_2$, ($\mathcal{R}_2$ is applied first, and then $\mathcal{R}_1$).

(b) $\mathcal{R}_2 \circ \mathcal{R}_1$, ($\mathcal{R}_1$ is applied first, and then $\mathcal{R}_2$).

10. A relation is defined by the following table:

	1	2	3
1	×	×	
2		×	
3			×

Verify that this relation is not symmetric. Furthermore, show that we can add one element so that the resulting relation is symmetric.

11. A relation is defined by the following table:

	1	2	3
1	×		×
2	×	×	×
3	×	×	×

Verify that this relation is not transitive. Furthermore, show that we can remove one or more elements so that the resulting relation is transitive,

12. A relation is defined by the following table:

	1	2	3
1		×	
2	×	×	×
3			×

Show that this relation is not reflexive. What element should you add so that the resulting relation is reflexive?

13. Assume $\mathcal{R}$ is a reflexive relation on a set A. Prove that $\mathcal{R}$ is an equivalence relation on A if and only if $\forall a, b, c \in A$, $(a, b), (a, c) \in \mathcal{R}$ implies that $(b, c) \in \mathcal{R}$.

Solutions:.

1. Fill in the following definitions considering that A is a nonempty set:

   - A **relation on** A is defined as a subset of $A \times A$.
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be **symmetric** if $\forall x \in A \forall y \in A\,((x, y) \in \mathcal{R} \rightarrow (y, x) \in \mathcal{R})$.
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be **transitive** if $\forall x \in A \forall y \in A \forall z \in A((x, y) \in \mathcal{R} \wedge (y, z) \in \mathcal{R} \rightarrow (x, z) \in \mathcal{R})$.
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be **reflexive** if $\forall x \in A((x, x) \in \mathcal{R})$.
   - If $\mathcal{R}$ is a relation on set A, then $\mathcal{R}$ is said to be an **equivalence relation** if it is reflexive, symmetric and transitive.
   - If $\mathcal{R}_1$ and $\mathcal{R}_2$ are relations on A, then $\mathcal{R}_2 \circ \mathcal{R}_1$ represents the composition of $\mathcal{R}_1$ and $\mathcal{R}_2$, and is defined by

     $\mathcal{R}_2 \circ \mathcal{R}_1 = \{\ (x, y) \mid \exists t \in A$ so that $(x, t) \in \mathcal{R}_1$ and $(t, y) \in \mathcal{R}_2\ \}$.

2. Check whether the following statements are true or false. Justify your answer briefly.

   (a) If a relation is symmetric, then it is not antisymmetric.

   False. Consider $\mathcal{R} = \{(1, 1), (2, 2), (3, 3)\}$ on $A = \{1, 2, 3\}$. Then $\mathcal{R}$ is symmetric and it is also antisymmetric. In general, any relations containing only the elements in the form (a, a) are both symmetric and antisymmetric. Such a relation is called a **diagonal** relation on A.

   (b) A relation is either reflexive or irreflexive.

   False. Let $\mathcal{R} = \{(0, 0), (1, 1), (2, 4)\}$ be a relation on $A = \{0, 1, 2, 3, 4\}$. Then $\mathcal{R}$ is neither reflexive nor irreflexive, since $(3, 3) \notin \mathcal{R}$ and $(0, 0) \in \mathcal{R}$.

   (c) If A is a nonempty set, then any relation on A represents a function $f : A \rightarrow A$.

   False. Let $\mathcal{R} = \{(1, 1), (1, 2)\}$ be a relation on the set $A = \{1, 2\}$. Then $\mathcal{R}$ does not represent a function, since the domain value 1 has two codomain values assigned, whereas the domain value 2 has no value assigned. Either of these two facts would be enough to deny that $\mathcal{R}$ represents a function.
   Note that the converse of this statement is true, that is, any function on A represents a relation on A.

   (d) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are reflexive on a set A, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is reflexive as well.

   True. Since for all $x \in A$, by the reflexivity of both $\mathcal{R}_1$ and $\mathcal{R}_2$, $(x, x) \in \mathcal{R}_1$ and $(x, x) \in \mathcal{R}_2$, and by the definition of composition of the relations, $(x, x) \in \mathcal{R}_1 \circ \mathcal{R}_2$.

   (e) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are transitive on a set A, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is transitive as well.

   False. Let $\mathcal{R}_1 = \{(2, 3), (4, 5)\}$ and $\mathcal{R}_2 = \{(1, 2), (3, 4)\}$ be two transitive relations defined on the set $A = \{1, 2, 3, 4, 5\}$. Then $\mathcal{R}_1 \circ \mathcal{R}_2 = \{(1, 3), (3, 5)\}$ is not transitive, since $(1, 3) \in \mathcal{R}_1 \circ \mathcal{R}_2$ and $(3, 5) \in \mathcal{R}_1 \circ \mathcal{R}_2$, but $(1, 5) \notin \mathcal{R}_1 \circ \mathcal{R}_2$.

(f) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are transitive, then $\mathcal{R}_1 \cup \mathcal{R}_2$ is transitive as well.

False. Let the two transitive relations $\mathcal{R}_1 = \{(1,2),(3,4)\}$ and $\mathcal{R}_2 = \{(2,1),(3,4)\}$ be defined on the set $A = \{1,2,3,4\}$. Then $\mathcal{R}_1 \cup \mathcal{R}_2 = \{(1,2),(3,4),(2,1)\}$ is not transitive, since $(1,2) \in \mathcal{R}_1 \cup \mathcal{R}_2$ and $(2,1) \in \mathcal{R}_1 \cup \mathcal{R}_2$, but $(1,1) \notin \mathcal{R}_1 \cup \mathcal{R}_2$.

(g) If two relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are reflexive on a set A, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive on A as well.

True. For all $x \in A$, $(x,x) \in \mathcal{R}_1$ and $(x,x) \in \mathcal{R}_2$, since $\mathcal{R}_1$ and $\mathcal{R}_2$ are reflexive. Hence $(x,x) \in \mathcal{R}_1 \cap \mathcal{R}_2$ by the definition of intersection.

3. Determine if the relation is symmetric, antisymmetric, reflexive, irreflexive, or transitive on the set $A = \{1,2,3,4\}$.

(a) $\mathcal{R}_1 = \{(1,1),(2,3),(2,4),(3,2),(3,4),(4,4)\}$.

$\mathcal{R}_1$ is not symmetric since $(2,4) \in \mathcal{R}_1$, but $(4,2) \notin \mathcal{R}_1$.
$\mathcal{R}_1$ is not antisymmetric since $(2,3) \in \mathcal{R}_1$ and $(3,2) \in \mathcal{R}_1$, but $3 \neq 2$.
$\mathcal{R}_1$ is not reflexive since $(2,2) \notin \mathcal{R}_1$.
$\mathcal{R}_1$ is not irreflexive since $(1,1) \in \mathcal{R}_1$.
$\mathcal{R}_1$ is not transitive since $(2,3) \in \mathcal{R}_1$ and $(3,2) \in \mathcal{R}_1$, but $(2,2) \notin \mathcal{R}_1$.

(b) $\mathcal{R}_2 = \{(1,1),(1,2),(1,3),(2,1),(3,1),(4,4)\}$.

$\mathcal{R}_2$ is symmetric.
$\mathcal{R}_2$ is not anti-symmetric since $(1,2)$ and $(2,1)$ $\mathcal{R}_2$, but $1 \neq 2$.
$\mathcal{R}_2$ is not reflexive since $(2,2) \notin \mathcal{R}_2$.
$\mathcal{R}_2$ is not irreflexive since $(1,1) \in \mathcal{R}_2$.
$\mathcal{R}_2$ is transitive.

4. Given the following relations, check whether they are equivalence relations:

(a) x is logically equivalent to y, on the set of all propositions.

"Two propositions are logically equivalent if and only if they share the same truth table."

This relation is an equivalence relation because the reflexivity, symmetry, and the transitivity properties are true by the definition of equivalent propositions given above.

- Reflexive: For any proposition x, x is logically equivalent to itself.
- Symmetric: If x has the same truth table as y, then y has the same truth table as x. Thus, for any two propositions x, y, if x is logically equivalent to y, then y is logically equivalent to x.
- Transitive: If x has the same truth table as y and y has the same truth table as z, then x has the same truth table as z. Thus, for any three propositions x, y, z, if x is logically equivalent to y and y is logically equivalent to z, then x is logically equivalent to z.

(b) $x \subseteq y$, on the set of all subsets of the real numbers $\mathbb{R}$.

It is not an equivalence relation, as it is not symmetric: $\mathbb{Z} \subseteq \mathbb{Q}$, but $\mathbb{Q} \not\subseteq \mathbb{Z}$.

(c) $f(x)$ is big-$\mathcal{O}$ of $g(x)$ on the set of real valued functions on the domain of real numbers.

It is not an equivalence relation as it is not symmetric: x^2 is big-$\mathcal{O}$ of x^5, but x^5 in NOT big-$\mathcal{O}$ of x^2.

(d) $f(x)$ is big-Θ of $g(x)$ on the set of real valued functions on the domain of real numbers.

Intuitively, $f(x)$ is big-Θ of $g(x)$ if f and g have the same asymptotic (as the input values approach infinity) growth rate. Using this intuitive interpretation, we can convince ourselves that a big-Θ relation is an equivalence relation.

- Reflexive: For any function f, $f(x)$ is big-Θ of $f(x)$. Any function is big-Θ of itself.
- Symmetric: If f for any two functions f, g, if $f(x)$ is big-Θ of $g(x)$, then $g(x)$ is big-Θ of $f(x)$.
- Transitive: For any three functions f, g, h, if $f(x)$ is big-Θ of $g(x)$ and $g(x)$ is big-Θ of $h(x)$, then $f(x)$ is big-Θ of $h(x)$.

  We omit a rigorous proof for this statement, but if you are interested in giving one, use the following theorem:

  "$f(x)$ is big-Θ of $g(x)$ if and only if there exist positive constants C_1, C_2 and k such that $0 < C_1 \leq \dfrac{|f(x)|}{|g(x)|} \leq C_2$ if $x > k$."

(e) x divides y on the set of non-zero integers.

It is not an equivalence relation as it is not symmetric: $5|190$, but $190 \nmid 5$.

(f) $x \equiv y \pmod 7$ on the set of integers.

It is an equivalence relation. We use the following definition to justify it.

"$x \equiv y \pmod 7$ if and only it $x - y$ is divisible by 7, i.e., $x - y = 7q$ for some integer q."

- Reflexive: For any integer x, $x - x = 0$ and 0 is divisible by 7. Thus, $x \equiv x \pmod 7$.
- Symmetric: If $x - y$ is divisible by 7, then $y - x = -(x - y)$ is also divisible by 7. Thus, for any two integers x, y, if $x \equiv y \pmod 7$, then $y \equiv x \pmod 7$.
- Transitive: Assume $x - y$ and $y - z$ are divisible by 7, i.e., $x - y = 7k$ and $y - z = 7l$ for some integers k, l. Then $x - z = (x - y) + (y - z) = 7k + 7l = 7(k + l)$, where $k + l$ is an integer, and hence, $x - z$ is divisible by 7. Thus, for any three integers x, y, z, if $x \equiv y \pmod 7$ and $y \equiv z \pmod 7$, then $x \equiv z \pmod 7$.

5. Given the following relations on the set of people, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

(a) x is taller than y.

- Not symmetric: For any two people x and y, if x is taller than y, then y is NOT taller than x.
- Antisymmetric: The conditional statement "x taller than y and y is taller than $x \to x = y$" is a true statement for all people x and y, since the premise of the conditional statement is false.

- Not reflexive: A person is not taller than themself.
- Irreflexive: No one is taller than themself.
- Transitive: If person x is taller than person y and person y is taller than person z, then person x is taller than person z.

(b) x is born the same year as y.

- Symmetric: For any two people x and y, if x is born the same year as y, then y is born the same year as x.
- Not antisymmetric: There are two different people who were born the same year.
- Reflexive: For any person x, x is born the same year as x.
- Not irreflexive: If a relation is reflexive, then it cannot be irreflexive.
- Transitive: If person x and person y are born same year and person y and person z are born the same year, then person x is born the same year as person y.
- It is an equivalence relation.

(c) x and y have a common grandparent.

- Symmetric: For any two people x and y, if x and y have a common grandparent, then y and x have a common grandparent.
- Not antisymmetric: Two different people, such as siblings, can have a common grandparent.
- Reflexive: A person has the same grandparent as themself.
- Not irreflexive: If a relation is reflexive, then it cannot be irreflexive.
- Not transitive: Let us assume that person x and person y are cousins, their mothers being sisters, so they have a common grandparent. Furthermore let us assume that person y and person z are also cousins, their fathers being brothers, so they also share a grandparent. Then person x and person z don't necessarily have a common grandparent.

6. Given the following relations on the set of integers, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

(a) x and y have a common prime factor.

- Symmetric: If x and y have a common prime factor, then so do y and x.
- Not antisymmetric: 6 and 9 have a common prime factor but $6 \neq 9$.
- Not reflexive: 1 is not related to itself, since 1 has no prime factors. 1 is the only integer which is not related to itself.

- Not irreflexive: 12 and 12 have the common prime factor 3. Every integer but 1 is related to itself.
- Not transitive: 12 and 15 have the common prime factor 3, and 15 and 125 have the common prime factor 5, but 12 and 125 have no common prime factors.

(b) $x - y$ is a multiple of 5.

- Symmetric: If $5|(x-y)$ then $5|(y-x)$, as $y - x = -(x-y)$ for any integers x and y.
- Not antisymmetric: $5|(23-8)$ and $5|(8-23)$ but $23 \neq 8$.
- Reflexive: Since $x - x = 0$, $5|(x-x)$ for any integer x.
- Not irreflexive: Since this relation is reflexive, it cannot be irreflexive.
- Transitive: If $5|(x-y)$, say $x - y = 5s$ for some integer s and $5|(y-z)$, say $y - z = 5t$ for some integer t, then $x - z = (x-y) + (y-z) = 5s + 5t = 5(s+t)$. Hence $5|(x-z)$.
- It is an equivalence relation.

7. Given the following relations on the set of positive integers, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

(a) $x - y = 1$.

- Not symmetric: $5 - 4 = 1$, but $4 - 5 = -1 \neq 1$
- Antisymmetric: The conditional statement "if $x - y = 1$ and $y - x = 1$, then $x = y$" is true for all positive integers x and y, since the premise is false.
- Not reflexive: $7 - 7 = 0 \neq 1$, thus 7 is not related to 7.
- Irreflexive: $x - x = 0 \neq 1$ for all positive integers x. Thus, there are no positive integers that are related to themselves.
- Not transitive: $6 - 5 = 1$ and $5 - 4 = 1$, but $6 - 4 = 2 \neq 1$.

(b) $|x - y| \geq 1$.

- Symmetric: For all positive integers x and y, $|x-y| = |y-x|$.
- Not antisymmetric: $|9-3| \geq 1$ and $|3-9| \geq 1$, but $9 \neq 3$.
- Not reflexive $|7-7| = 0 \ngeq 1$, so 7 is not related to 7.
- Irreflexive: $|x-x| = 0 \ngeq 1$, for all positive integers x.
- Not transitive: $|9-3| = 6 \geq 1$ and $|3-9| = 6 \geq 1$, but $|9-9| = 0 \ngeq 1$.

8. Given the following relations on the set of real numbers, determine if they have any of the properties: symmetric, antisymmetric, reflexive, irreflexive, and transitive.

   (a) $x - y$ is a rational number.

   - Symmetric: If $x - y$ is a rational number, then $y - x = -(x - y)$ and the opposite of a rational number is also rational.
   - Not antisymmetric: $4 - 3$ and $3 - 4$ are both rational, but $3 \neq 4$.
   - Reflexive: $x - x = 0$ is rational for all real numbers x.
   - Not irreflexive: A reflexive relation cannot be irreflexive.
   - Transitive: If $x - y = r_1$ and $y - z = r_2$ are rational, then $x - z = (x - y) + (y - z) = r_1 + r_2$ is rational as well, since the sum of rational numbers is rational.

   (b) $x \cdot y \geq 0$.

   - Symmetric: If $x \cdot y \geq 0$, then $y \cdot x \geq 0$, since $y \cdot x = x \cdot y$
   - Not antisymmetric: $2 \cdot 3 \geq 0$ and $3 \cdot 2 \geq 0$ but $2 \neq 3$.
   - Reflexive: For all real numbers x, $x^2 \geq 0$.
   - Not irreflexive: A reflexive relation cannot be irreflexive.
   - Transitive: If $x \cdot y \geq 0$ and $y \cdot z \geq 0$, then either all three numbers $x, y, z \geq 0$, or all three $x, y, z \leq 0$. In both cases $x \cdot z \geq 0$ which proves the transitivity.

9. Given two relations on the set $\{1, 2, 3\}$ by the following tables:

$\mathcal{R}_1$:

	1	2	3
1	×		
2		×	×
3	×		

$\mathcal{R}_2$:

	1	2	3
1			×
2	×	×	
3			×

construct the tables for

   (a) $\mathcal{R}_1 \circ \mathcal{R}_2$, ($\mathcal{R}_2$ is applied first, and then $\mathcal{R}_1$).

$\mathcal{R}_1 \circ \mathcal{R}_2 =$

	1	2	3
1	×		
2	×	×	×
3	×		

   (b) $\mathcal{R}_2 \circ \mathcal{R}_1$, ($\mathcal{R}_1$ is applied first, and then $\mathcal{R}_2$).

$\mathcal{R}_2 \circ \mathcal{R}_1 =$

	1	2	3
1			×
2	×	×	×
3			×

10. A relation is defined by the following table:

	1	2	3
1	×	×	
2		×	
3			×

Verify that this relation is not symmetric. Furthermore, show that we can add one element so that the resulting relation is symmetric.

The relation is not symmetric since $(1, 2) \in \mathcal{R}$, but $(2, 1) \notin \mathcal{R}$. To obtain a symmetric relation, we have to add the element $(2, 1)$.

Symmetry means that the crosses must be symmetric with respect to the diagonal. The diagonal squares may or may not be checked.

11. A relation is defined by the following table:

	1	2	3
1	×		×
2	×	×	×
3	×	×	×

Verify that this relation is not transitive. Furthermore, show that we can remove one or more elements so that the resulting relation is transitive,

The relation is not transitive since $(1, 3) \in \mathcal{R}$ and $(3, 2) \in \mathcal{R}$, but $(1, 2) \notin \mathcal{R}$. To obtain a transitive relation, we have to remove the element $(3, 2)$.

There is no good visualization for transitivity.

12. A relation is defined by the following table:

	1	2	3
1		×	
2	×	×	×
3			×

Verify that this relation is not reflexive. What element should you add so that the resulting relation is reflexive?

The relation is not reflexive since $(1, 1) \notin \mathcal{R}$. To obtain a reflexive relation, we have to add the element $(1, 1)$.

Reflexivity means that the diagonal elements in the table are completely filled with crosses.

13. Assume $\mathcal{R}$ is a reflexive relation on a set A. Prove that $\mathcal{R}$ is an equivalence relation on A if and only if $\forall a, b, c \in A$, $(a, b), (a, c) \in \mathcal{R}$ implies that $(b, c) \in \mathcal{R}$.

Proof. Let $\mathcal{R}$ be a reflexive relation on a set A.

First we are going to prove that if $\mathcal{R}$ is an equivalence relation, then $\forall a, b, c \in A$, $(a, b), (a, c) \in \mathcal{R}$ implies that $(b, c) \in \mathcal{R}$.

If $\mathcal{R}$ is an equivalence relation, then, in addition of being reflexive, it is also symmetric and transitive. Assume $(a, b), (a, c) \in \mathcal{R}$. By symmetry of $\mathcal{R}$, from $(a, b) \in \mathcal{R}$ we infer $(b, a) \in \mathcal{R}$. Applying the transitivity to $(b, a), (a, c) \in \mathcal{R}$ we obtain $(b, c) \in \mathcal{R}$.

Now we show that if $\forall a, b, c \in A$, $(a, b), (a, c) \in \mathcal{R}$ implies that $(b, c) \in \mathcal{R}$, then $\mathcal{R}$ is an equivalence relation on the set A.

Let us assume that $\mathcal{R}$ has the property that $\forall a, b, c \in A$, $(a, b), (a, c) \in \mathcal{R}$ implies that $(b, c) \in \mathcal{R}$. To prove that $\mathcal{R}$ is symmetric, assume $(a, b) \in \mathcal{R}$. Since $\mathcal{R}$ is reflexive, $(a, a) \in \mathcal{R}$, so from the given property of $\mathcal{R}$, we infer that $(b, a) \in \mathcal{R}$, hence $\mathcal{R}$ is symmetric. To prove that $\mathcal{R}$ is transitive, assume (a, b) and $(b, c) \in \mathcal{R}$. By symmetry, from $(a, b) \in \mathcal{R}$ we get that $(b, a) \in \mathcal{R}$ and now using the given property of $\mathcal{R}$, we can infer that $(a, c) \in \mathcal{R}$, hence $\mathcal{R}$ is transitive. Since $\mathcal{R}$ was given to be reflexive, at this point we already showed that $\mathcal{R}$ is an equivalence relation. ■