Concepts:

- Outline a proof for the Inclusion-Exclusion Principle for two or more sets.
- Evaluate or bound the cardinality of a union of sets.
- Evaluate or bound the cardinality of an intersection of two sets.
- Solve counting problems involving the Inclusion-Exclusion Principle.
- Use a Venn diagram to visualize the Inclusion-Exclusion Principle.

Problems:

1. Suppose A and B are two sets. No additional information is available. What is the maximum possible value of $|A \cup B|$?

2. The intersection of two sets contains 10 elements, and the union 45. What is the sum of the number of elements in the two sets?

3. Suppose A and B are two sets such that A contains 12 elements and B contains 8 and both A and B have 5 elements in common. How many elements are in $A \cup B$?

4. At a pet owner's meeting of 150 people, there are exactly 67 dog owners, and 56 cat owners. Some people may own neither and some people may own both animals.

   (a) What is the maximum number of people who could have both, a cat and a dog?

   (b) What is the minimum number of people who could have both, a cat and a dog?

5. At a pet owner's meeting of 150 people, there are exactly 67 dog owners, and 56 cat owners. A sociologist wants to interview at least 10 people with unusual pets, which she defines as any pet that is neither a cat nor a dog. At least how many people does she need to select at random to interview to guarantee 10 people with unusual pets?

6. At College X, 130 students were surveyed about their electronic possessions.

   90 had a cell phone. 50 had an iPod. 80 had a computer. 70 had a cell phone and a computer. 20 had a cell phone and an iPod. 16 had a computer and an iPod. 13 had a cell phone, an iPod, and a computer. How many students had at least one of the devices?

7. Find the number of positive integers ≤ 1000 that are multiples of at least one of the integers 4, 6, and 15.

8. Seven sets, all with the same number of elements, n. There are no elements common to all sets. There is 1 common to any six of them, 2 common to any five, 3 common to any four, 4 common to any three and 6 common to any pair of sets. The union of the seven sets has 28 elements. Find n.

9. How many 5-digit numbers do not start with a 3, do not end with two zeros and are not even?

10. How many 5-card poker hands contain the ace of spades or only black cards or only spades?

11. Permute the letters in the English alphabet forming a long string of letters (of length 26, each letter is used exactly once.) How many different such permutation does not contain any of the words *great*, *job* or *quiz*. Hint: find the total number of such permutations minus the ones that do contain at least one of those words using the Inclusion-Exclusion Principle.

Challenge Problems:

12. Find the number of surjective functions from a set with 7 elements to a set with 4 elements.

13. (University of South Carolina High School Math Contest, 1993.) Suppose that 4 cards labeled 1 to 4 are randomly placed in 4 boxes also labeled 1 to 4, one card per box. What is the probability that no card is placed in a box that has the same label as the card?

Solutions:

1. Suppose A and B are sets. No additional information is available. What is the maximum possible value of $|A \cup B|$?

   By the Inclusion Exclusion Principle $|A \cup B| = |A| + |B| - |A \cap B|$. Thus, $|A \cup B| \leq |A| + |B|$ for all sets A and B. Furthermore, $|A \cup B| = |A| + |B|$ if and only if A and B are disjoint, that is, $|A \cap B| = \emptyset$.

2. The intersection of two sets contains 10 elements, and the union 45. What is the sum of the number of elements in the two sets?

   By the Inclusion-Exclusion Principle $|A| + |B| = |A \cup B| + |A \cap B| = 45 + 10 = 55$.

3. A and B are sets. A contains 12 elements and B contains 8. A and B have 5 elements in common. How many elements are in $A \cup B$?

   Using the formula $|A \cup B| = |A| + |B| - |A \cap B| = 12 + 8 - 5 = 15$.

4. At a pet owner's meeting of 150 people, there are exactly 67 dog owners, and 56 cat owners. Some people may own neither and some people may own both animals.

   (a) What is the maximum number of people who could have both, a cat and a dog?

   (b) What is the minimum number of people who could have both, a cat and a dog?

   Let D be the set of people at this meeting owning dogs, and C be the set of people at this meeting owning cats. Let $\mathcal{U}$ be the set of all people at this meeting, which is the universal set. The given information is $|D| = 67, |C| = 56$, and $|\mathcal{U}| = 150$.

   (a) Since $C \cap D \subseteq C$ and $C \cap D \subseteq D$, $|C \cap D| \leq |C| = 56$. Hence, the maximum number of people who could have both a cat and a dog is 56. In this case, all cat owners own a dog.

   (b) The minimum number of elements in the intersection $C \cap D$ happens when $C \cap D = \emptyset$, if that is possible. Here it is possible, since $|C| + |D| < |\mathcal{U}|$. Hence, the minimum number of people at this meeting who could own both a cat and a dog is 0.

5. At a pet owner's meeting of 150 people, there are exactly 67 dog owners, and 56 cat owners. A sociologist wants to interview at least 10 people with unusual pets, which she defines as any pet that is neither a cat nor a dog. At least how many people does she need to select at random to interview to guarantee 10 people with unusual pets?

   Let D be the set of people at this meeting owning dogs, and C be the set of people at this meeting owning cats. Let $\mathcal{U}$ be the set of all people at this meeting, hence the universe. The given information is $|D| = 67, |C| = 56$, and $|\mathcal{U}| = 150$.

   The set of "usual pets" owners is $C \cup D$ and the set of unusual pet owners is its complement. To be sure we end up with 10 from the latter group, we have to choose "the largest possible number of usual pets owners" + 10, or $|C| + |D| + 10 = 56 + 67 + 10 = 133$. If we select only 132 pet owners, it is possible that nobody owns both cats and dogs, and in that case, there would only be 9 people with unusual pets.

   Note. The fact that there are 150 people at this meeting and $150 > 133$ makes the problem possible.

6. At College X, 130 students were surveyed about their electronic possessions.

   90 had a cell phone. 50 had an iPod. 80 had a computer. 70 had a cell phone and a computer. 20 had a cell phone and an iPod. 16 had a computer and an iPod. 13 had a cell phone, an iPod, and a computer.

   How many students had at least one of the devices?

   Denote the set of students who own a cell phone by P, the set of students owning an iPod by I, and the set of students owning a computer by C. We need to calculate $|P \cup I \cup C|$. We use the Inclusion-Exclusion Principle for three sets. Hence,

   $|P \cup I \cup C| = |P| + |I| + |C| - |P \cap I| - |P \cap C| - |I \cap C| + |P \cap I \cap C|$

   $= 90 + 50 + 80 - 20 - 70 - 16 + 13 = 127$

   Note. Use a Venn diagram with three sets to visualize the given sets and their intersections.

7. Find the number of positive integers ≤ 1000 that are multiples of at least one of the integers 4, 6, and 15.

   Note that the set to consider and determine the cardinality of can be expressed as $S_1 \cup S_2 \cup S_3$, where

   S_1 is the set of positive integers ≤ 1000 that are multiples of 4,
   S_2 is the set of positive integers ≤ 1000 that are multiples of 6, and
   S_3 is the set of positive integers ≤ 1000 that are multiples of 15.

   By the Inclusion-Exclusion Principle,

   $$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.$$

   Now we calculate each summand in the above formula.

   $|S_1| = 250$, since $1000/4 = 250$. The elements of S_1 are $4 \cdot 1, 4 \cdot 2, 4 \cdot 3, \ldots, 4 \cdot 250$.
   $|S_2| = 166$, since $1000/6 = 166.666$. The elements of S_2 are $6 \cdot 1, 6 \cdot 2, 6 \cdot 3, \ldots, 6 \cdot 166$.
   $|S_3| = 66$, since $1000/15 = 66.666$. The elements of S_2 are $15 \cdot 1, 15 \cdot 2, 15 \cdot 3, \ldots, 15 \cdot 66$.

   $S_1 \cap S_2 =$ is the set of all positive integers ≤ 1000 which are multiples of 12, so $|S_1 \cap S_2| = 83$.
   $S_1 \cap S_3 =$ the set of all positive integers ≤ 1000 which are multiples of 60, so $|S_1 \cap S_3| = 16$.
   $S_2 \cap S_3 =$ the set of all positive integers ≤ 1000 which are multiples of 30, so $|S_2 \cap S_3| = 33$, and
   $S_1 \cap S_2 \cap S_3 =$ the set of all positive integers ≤ 1000 which are multiples of 60, so $|S_1 \cap S_2 \cap S_3| = 16$.

   Hence, the number of positive integers ≤ 1000 that are multiples of at least one of the integers 4, 6, and 15 is equal to

   $250 + 166 + 66 - (83 + 16 + 33) + 16 = 366.$

8. Consider seven sets, all with the same number of elements, n. There are no elements common to all sets. There are 1 common to any six of them, 2 common to any five, 3 common to any four, 4 common to any three, and 6 common to any pair of sets. The union of the seven sets has 28 elements. Find n.

   Use the Inclusion-Exclusion Principle to obtain the following:

$28 = C(7,1)n - C(7,2)\cdot 6 + C(7,3)\cdot 4 - C(7,4)\cdot 3 + C(7,5)\cdot 2 - C(7,6)\cdot 1$

Hence, $28 = 7n - 21\cdot 6 + 35\cdot 4 - 35\cdot 3 + 21\cdot 2 - 7\cdot 1$.

Solving the resulting linear equation in n, we get the solution $n = 12$.

9. How many 5-digit numbers do not start with a 3, do not end with two zeros and are not even?

Let $\mathcal{U}$ denote all sets of 5-digit numbers. Then $|\mathcal{U}| = 9\cdot 10^4$ because the first digit can be $1-9$, all other digits can be any of $0-9$.

A is the set of elements in $\mathcal{U}$ that start with 3. Then $|A| = 10^4$ because except for the first digit, all other digits can be any of $0-9$.

B is the set of elements in $\mathcal{U}$ that end with two zeros. Then $|B| = 9\cdot 10^2$ because the first digit can not be 0, the next two digits can be $0-9$ and the last two digits are zeros.

C is the set of elements in $\mathcal{U}$ that are even. Then $|C| = 9\cdot 10^3\cdot 5$ because the first digit cannot be 0, the next three digits can be $0-9$ and the last digit can be $0,2,4,6,8$.

Then, the number of 5-digit numbers that do not start with a 3, do not end with two zeros, and are not even is $\mathcal{U}\setminus(A\cup B\cup C)$.

By the Inclusion-Exclusion Principle $|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$.

Now we calculate the cardinalities of the intersections in the formula:

The set $A\cap B$ contains all five-digit numbers that start with a 3 and end with two zeros, and hence $|A\cap B| = 10^2$.

The set $A\cap C$ contains all 5-digits numbers that start with a 3 and have the last digit $0,2,4,6,8$, and hence $|A\cap C| = 10^3\cdot 5$.

The set $B\cap C$ contains all 5-digits numbers that end with two zeros and are even, and hence $|B\cap C| = 9\cdot 10^2$. Note that $B\subseteq C$.

The set $A\cap B\cap C$ contains all five-digit numbers that start with a 3, end with two zeros, and are even, and hence $|A\cap B\cap C| = |A\cap B| = 10^2$.

Consequently, $|A\cup B\cup C| = 10^4 + 9\cdot 10^2 + 9\cdot 10^3\cdot 5 - 10^2 - 10^3\cdot 5 - 9\cdot 10^2 + 10^2 = 50000$, and

$|\mathcal{U}\setminus(A\cup B\cup C)| = |\mathcal{U}| - |A\cup B\cup C| = 90000 - 50000 = 40000$.

10. How many 5-card poker hands contain the ace of spades or only black cards or only spades?

Let A be the set of 5-card poker hands that contain the ace of spades. Then $|A| = C(51,4)$ because we choose the other 4 cards from the remaining 51 cards.

Let B be the set of 5-card poker hands that contain only black cards. Then $|B| = C(26,5)$ because all 5 cards are chosen from the 26 black cards.

Let C be be the set of 5-card poker hands containing only spades. Then $|C| = C(13, 5)$ because all 5 cards are chosen from the 13 spades.

We will need to use the Inclusion Exclusion Principle so we also need to find $|A \cap B|$, $|A \cap C|$, $|B \cap C|$ and $|A \cap B \cap C|$.

The set $A \cap B$ contains all 5-card poker hands that contain the ace of spades and only black cards, and hence $|A \cap B| = C(25, 4)$ because we choose the other 4 cards from the remaining 25 black cards.

The set $A \cap C$ contains all 5-card poker hands containing the ace of spades and only spades, and hence $|A \cap C| = C(12, 4)$ because we choose the other 4 cards from the remaining 12 spades.

The set $B \cap C$ contains all 5-card poker hands containing only black cards and only spades, and hence $|B \cap C| = |C| = C(13, 5)$.

The set $A \cap B \cap C$ contains all five card poker hands containing the ace of spades and only black cards and only spades, and hence $|A \cap B \cap C| = |A \cap C| = C(12, 4)$.

By the Inclusion-Exclusion Principle, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| =$

$C(51, 4) + C(26, 5) + C(13, 5) - C(25, 4) - C(12, 4) - C(13, 5) + C(12, 4) = 249900 + 65780 + 1287 - 12650 - 495 - 1287 + 495 = 303030$.

11. Permute the letters in the English alphabet forming a long string of letters (of length 26, each letter is used exactly once.) How many different such permutation does not contain any of the words *great*, *job* or *quiz*. Hint: find the total number of such permutations minus the ones that do contain at least one of those words using the Inclusion-Exclusion Principle.

    Let A be the set of all permutations that contain *great*,

    let B be the set of all permutations that contain *job*, and

    let C be the set of all permutations that contain *quiz*.

    Then $|A| = 22!$ as we order a total of 22 "letters", thinking of *great* as a super-letter and the other 21 as regular letters.

    Similarly $|B| = 24!$ as we order a total 24 "letters", thinking of *job* as a super-letter and the other 23 as regular letters, and

    $|C| = 23!$ as we order a total of 23 "letters", thinking of *quiz* as a super-letter and the other 22 as regular letters.

    The set $A \cap B$ contains all permutations that contain both *great* and *job*, thus $|A \cap B| = 20!$. Here we think of *great* and *job* as super-letters and the other 18 as regular letters, so we order a total of 20 "letters."

    Similarly $|A \cap C| = 19!$ and $|B \cap C| = 21!$

    Finally, the set $A \cap B \cap C$ contains all permutations that contain all three words *great*, *job* and *quiz*, thinking of 3 super-letters and the other 14 as regular letters. Thus, $|A \cap B \cap C| = 17!$.

    Applying the Inclusion-Exclusion Principle,

$|A_1 \cup A_2 \cup A_3| = 22! + 24! + 23! - 20! - 19! - 21! + 17!.$

Since the number of permutations that do not contain any of the three words is the total number of permutations 26! minus the ones that contain at least one of the permutations, the final answer is

$$26! - 22! - 24! - 23! + 20! + 19! + 21! - 17!$$

Challenge Problems:

12. Find the number of surjective functions from a set with 7 elements to a set with 4 elements.

We need to count the number of elements in the set $\{\ f \mid f : A \to B,\ f \text{ is surjective }\}$, where $|A| = 7$ and $|B| = 4$.

Note that $\{\ f \mid f : A \to B,\ f \text{ is surjective }\} = \{\ f \mid f : A \to B\ \} \setminus \{\ f \mid f : A \to B,\ f \text{ is not surjective}\}$.
Recall that $|\{\ f \mid f : A \to B\ \}| = 4^7$, hence we only need to calculate $|\{\ f \mid f : A \to B,\ f \text{ is not surjective }\}|$.

Let $B = \{b_1, b_2, b_3, b_4\}$ and $B_i = \{f \mid f : A \to B \setminus \{i\}\}$, $i = 1, 2, 3, 4$.
Then $\{\ f \mid f : A \to B,\ f \text{ is not surjective }\} = B_1 \cup B_2 \cup B_3 \cup B_4$.

By the Inclusion-Exclusion Principle,

$$|B_1 \cup B_2 \cup B_3 \cup B_4| = \sum_{i=1}^{4} |B_i| - \sum_{i,j \in \{1,2,3,4\}, i<j}^{4} |B_i \cap B_j| + \sum_{i,j,k \in \{1,2,3,4\}, i<j<k}^{4} |B_i \cap B_j \cap B_k| - |B_1 \cap B_2 \cap B_3 \cap B_4|.$$

Now, $|B_i| = 3^7$, for $i \in \{1, 2, 3, 4\}$,
$|B_i \cap B_j| = 2^7$, for $i, j \in \{1, 2, 3, 4\}, i < j$,
$|B_i \cap B_j \cap B_k| = 1^7 = 1$, for $i, j, k \in \{1, 2, 3, 4\}, i < j < k$, and
$|B_1 \cap B_2 \cap B_3 \cap B_4| = 0$.

Hence $|\{\ f \mid f : A \to B,\ f \text{ is surjective }\}| = 4^7 - \big(C(4,1) \cdot 3^7 - C(4,2) \cdot 2^7 + C(4,3) \cdot 1 - 0\big) = 8400$.

13. (University of South Carolina High School Math Contest, 1993.) Suppose that 4 cards labeled 1 to 4 are randomly placed into 4 boxes also labeled 1 to 4, one card per box. What is the probability that no card is placed in a box that has the same label as the card?

The probability that no card is placed into a box having the same label as the card is the quotient of the number of such placements of the cards divided by the total number of ways to place the 4 different cards into 4 different boxes. The denominator is just the number of permutations of four elements, that is $4! = 24$.

Let $A = \{1, 2, 3, 4\}$. In order to calculate the value of the numerator, we need to find the number of injective functions from A to itself which does not fix any of the points in the domain.

That is, we have to find $|\{f \mid f : A \to A, f \text{ injective}, f(i) \neq i \text{ for } i \in A\}|$, where domain represents the card numbers and the codomain represents the label of the boxes. So, if $f(i) = j$, that means the card-i is placed into box-j.

Let us define the sets $A_i = \{f \mid f : A \to A, f \text{ injective and } f(i) = i\}$ for $i \in A$.

Then $\{f \mid f : A \to A, f \text{ injective and } f(i) \neq i \text{ for } i \in A\} = \{f \mid f : A \to A, f \text{ injective }\} \setminus \bigcup_{i=1}^{4} A_i$.

Recall that $|\{f \mid f : A \to A, f \text{ injective}\}| = 4!$, so we need to calculate $\bigcup_{i=1}^{4} A_i$.

We use the Inclusion-Exclusion Principle to calculate $\bigcup_{i=1}^{4} A_i$:

$|\bigcup_{i=1}^{4} A_i| = |A_1|+|A_2|+|A_3|+|A_4|-(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)+$
$(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) - |A_1 \cap A_2 \cap A_3 \cap A_4|$.
Note that we could have use a sigma notation to express this sum as we did in the problem 12.

Now we need to calculate the following cardinalities:

$|A_i| = 3!$, $i \in A$,
$|A_i \cap A_j| = 2!$, $i, j \in A, i < j$,
$|A_i \cap A_j \cap A_k| = 1$, $i, j, k \in A, i < j < k$, and finally
$|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$.

Thus, $|\bigcup_{i=1}^{4} A_i| = \underbrace{|A_1| + |A_2| + |A_3| + |A_4|}_{=4\cdot 3!} - \underbrace{(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)}_{=6\cdot 2!} +$
$\underbrace{(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|)}_{=4\cdot 1} - \underbrace{|A_1 \cap A_2 \cap A_3 \cap A_4|}_{=1} =$
$4 \cdot 3! - 6 \cdot 2! + 4 \cdot 1 - 1 = 15$.

Hence $|\{f \mid f : A \to A, f \text{ injective }\} \setminus \bigcup_{i=1}^{4} A_i| = 4! - 15 = 9$, and so the probability we had to calculate is $\frac{9}{24} = \frac{3}{8}$.

Note. The permutations of a set with n elements that fixes no element is called a **derrangement** of n elements. In this problem we calculated the number of derrangements of 4 elements to be 9.