Concepts:

- Apply the Sum and Multiplication Principles for counting the ways of carrying out tasks.
- Apply the Pigeonhole Principle to guarantee the size of repetition based on the given sample size.
- Apply the Pigeonhole Principle to determine the sample size needed to guarantee repetition of traits.
- Identify "selection with repetition" scenarios.
- Identify "selection without repetition" scenarios as ordered (permutations) or unordered (combinations).
- Count the number of such selections using the formulas for permutations and combinations.
- Solve more elaborate counting problems involving all the principles above.

Problems:

1. Fill in the blanks for the following sentences.

   (a) The Multiplication Principle states that ...

   (b) The Addition Principle states that ...

   (c) The Pigeonhole Principle states that if $n + 1$ items are placed in n containers, then ...

   (d) The Generalized Pigeonhole Principle states that if m items are placed in n containers, $m > n > 0$, then ...

   (e) If there are n distinguishable objects and we are selecting k objects from them, then the number of ordered selections with repetition is ...

   (f) If there are n distinguishable objects and we are selecting k objects from them, then the number of ordered selections without repetition is ...

   (g) If there are n distinguishable objects and we are selecting k objects from them, then the number of unordered selections without repetition is...

2. How many 3 letter "words" can be created from the letters ABCDEFG when

   (a) repetition allowed?

   (b) repetition is not allowed?

   The "words" do not have to be meaningful.

3. How many bit strings (i.e., strings of 0's and 1's) of length 14

   (a) start with a 1 and end with 00?

(b) start with a 1 or end with 00?

(c) contain exactly four 1's?

(d) contain at most twelve 1's?

(e) contain at least ten 1's

(f) have the sum of bits equal to 12

4. Let function f map the bit strings of length 4 into the set $\{0, 1, 2, 3, 4\}$ where f assigns to each bit string of length 4 the number of 0 bits that the input string contains. For example: $f(0100) = 3$ and $f(0110) = 2$. Justify your answers.

   (a) Is f one-to-one?

   (b) Is f onto?

5. Find the number of

   (a) functions from a set with 5 elements to a set with 6 elements.

   (b) injective functions from a set with 5 elements to a set with 6 elements.

   (c) injective functions from a set with 6 elements to a set with 5 elements.

   (d) surjective functions from a set with 6 elements to a set with 6 elements.

6. From the word "ALGORITHM", in how many ways can a

   (a) three-letter word be selected?

   (b) six-letter word be selected assuming the first letter must be a T?

   (c) six-letter word be selected assuming the word contains the letter T?

   (d) six-letter word be selected and assuming the letters I and M have to be together in the word, either as IM or MI?

   Repetitions are not allowed in any of the selections, that is, one letter can be selected only once.

7. How may hexadecimal strings of length 4

   (a) do not have any repeated digits?

   (b) have at least one repeated digit?

8. A computer programming team has 13 members, seven computer scientists and six software engineers.

(a) How many groups of seven can be chosen to work on a project?

(b) How many groups of seven can be chosen to work on a project if there must be three computer scientists and four software engineers in the group?

(c) Suppose that there are two members of the team who refuse to work together. How many groups of seven can be chosen to work on a project making sure these two people are never in the same group?

9. There are ten points in plane, $P_1, P_2, P_3, \dots P_{10}$, such that no three of them are colinear.

   (a) How many distinct lines are determined by these ten points?

   (b) How many triangles are formed by these ten points?

10. Three officers - a president, a treasurer, and a secretary - are to be chosen from among four people: Ann, Bob, Chandra and Diego.

    (a) How many ways can a president, a treasurer, and a secretary be selected from these four people?

    (b) Suppose that Bob is not qualified to be treasurer and Diego's other commitments make it impossible for him to be secretary. How many ways can a president, a treasurer, and a secretary be chosen from the given group of four people?

11. True or False? Justify your answer.

    (a) If there are 22 people in a room, then it is certain that at least 4 of them were born on the same day of the week.

    (b) In a group of 700 people at least 2 have the same first and last initials.

12. In a class of 3000 students, the professor gives a multiple choice quiz with 4 questions with possible answers A, B, C, D or E. Each student answers each question. Among the 3000 students, at least how many students will turn in the exact same answers for the quiz questions?

13. What is the minimum number of people you need to guarantee that there are three among them who were born on the same day? Recall there are 366 possible birthdays.

14. There are 193 member states of the United Nations. If you have 195 diplomats from these states in a room, then which of the following statements can be inferred?

    1. It is certain that at least 2 of them are from the same state.

    2. It is certain that at least 3 of them are from the same state.

15. A device produces random 64-bit integers at a rate of one billion per second. After how many years of running is it unavoidable that the device produces an output for the second time? Round to the nearest number of years.

16. A penny collection contains twelve pennies from the year 1967, seven pennies from the year 1968, and eleven pennies from the year 1971. If you were to pick some pennies without looking at the dates, how many must you pick to be sure of getting at least five pennies from the same year?

17. A bag contains 100 apples, 100 bananas, 100 oranges and 100 pears. If someone randomly picks a fruit from the bag every second,

    (a) how long will it take to be sure to have at least a dozen of the same kind of fruit?

    (b) how long will it take to be sure to have at least a dozen bananas?

18. A restaurant serves a 3-course lunch option for tourists. The tourists could choose between the following two options:

    Option 1: a soup, a main course and a desert,

    Option 2: an appetizer, a soup and a main course.

    There are 4 options for appetizers, 5 options for soup, 10 options for the main course and 3 different strudel options for dessert.

    What is the smallest number of tourists that have to visit this restaurant to guarantee that at least 10 tourists ordered the same 3-course lunch option?

19. An ice cream store gives the following options for the customers to create their own ice cream sundaes:

    1. Pick 3 different (can't select a flavour twice) ice cream flavours out of 8.
    2. Pick 2 different (no repetition) toppings out of 5.

    How many times does Helena have to visit this ice cream store so that it is unavoidable that she would eat the same sundae for the third time?

20. A password generator creates passwords which consist of a minimum of 6 and a maximum of 8 characters. The generator can use either digits (0-9), lower case letters, upper case letters, or the characters !? (exclamation and question marks). How many passwords need to be generated until it is unavoidable that the same password is created for the second time? Leave your answer in unevaluated form.

21. **Counting with Recursion.** Let a_n be the number of bit strings of length n which contain exactly two 0's.

    (a) Find a_1.

    (b) Find the recursive formula for $a_n, n \geq 2$.

    (c) Use elementary counting method to find a closed-form formula for a_n.

    (d) Use induction to show that $\{a_n\}$ defined recursively by $a_n = a_{n-1} + (n-1)$, $n \geq 2$, with $a_1 = 0$ is equivalent to the closed formula $a_n = \dfrac{n(n-1)}{2}$ for all positive integers n.

22. **Counting in Complexity Analysis.** Counting has many applications in computer science. We use systematic counting techniques to count the number of operations used by an algorithm to analyze time complexity. This problem asks you analyze a simple Python code segment.

    The range() function returns a list of integers, starting from 0 and increments by 1 and ends at a certain number. See the following examples:

    range(7) returns the list of integers 0, 1, 2, 3, 4, 5, 6 (not including 7).

    range(4,8) returns the list 4, 5, 6, 7 (not including 8).

(a) Count the number of additions and multiplications in the following Python code segment:

```
for i in range(26):
    for j in range(34,76):
        k=i+j+3
        l=2*i+3*j
        m=i*j+1
print(2*k+l+m)
```

(b) Count the number of additions and multiplications in the following Python code segment:

```
value=0
for x in range(50):
    for y in range(57):
        if y>x:
            value=value+y
print(value)
```

Solutions:

1. Fill in the blanks for the following sentences.

   (a) The Multiplication Principle states that if a procedure can be broken down into a sequence of two tasks, and if there are m ways of completing the first task, and for each outcome of the first task there are n ways of completing the second task independently of the outcome of the first task, then there are $m \cdot n$ different ways of completing the procedure.

   (b) The Addition Principle states that if in a procedure you can choose between two exclusive alternatives, either A or B, where the first alternative A can be completed in m ways and the second alternative B can be completed in n ways, then there are $m + n$ different ways of completing the procedure.

   (c) The Pigeonhole Principle states that if $n + 1$ items are placed into n containers, then at least one container will contain at least two items.

   (d) The Generalized Pigeonhole Principle states that if m items are placed in n containers, $m > n > 0$, then at least one container will contain at least $\left\lceil \frac{m}{n} \right\rceil$ items.

   (e) If there are n distinguishable objects and we are selecting k objects from them, then the number of ordered selections with repetition is n^k.

   (f) If there are n distinguishable objects and we are selecting k objects from them, then the number of ordered selections without repetition is $P(n, k) = n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$, the permutations of n objects taken k at a time.

   (g) If there are n distinguishable objects and we are selecting k objects from them, then the number of unordered selections without repetition is $C(n, k) = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \dfrac{n!}{(n-k)!k!}$, the combinations of n objects taken k at a time.

2. How many 3 letter "words" can be created from the letters ABCDEFG when

   (a) repetition allowed?

   We can use 7 letters, with repetition, to fill 3 positions independently. Hence there are $7 \cdot 7 \cdot 7 = 343$ "words". Note that this is an ordered selection with repetition.

   (b) repetition is not allowed?

   We can use 7 letters, without repetition, to fill 3 positions. This is an ordered selection without repetition, i.e., the permutation of 7 objects taken 3 at a time. Thus, there are $P(7, 3)$ OR $7 \cdot 6 \cdot 5$ such "words."

   The "words" do not have to be meaningful.

3. How many bit strings (i.e., strings of 0's and 1's) of length 14

(a) start with a 1 and end with 00?

Since three of the 14 positions of the strings are preset, we have only 11 free spaces to fill with 0's or 1's. Hence there are 2^{11} such bit-strings.

(b) start with a 1 or end with 00?

Let us define the following sets:

$A = \{s|s$ is a bit string of length 14 which starts with a 1 $\}$ and
$B = \{s|s$ is a bit string of length 14 which ends with 00 $\}$.

By the Inclusion-Exclusion Principle $|A \cup B| = |A| + |B| - |A \cap B|$ where $|A| = 2^{13}$, $|B| = 2^{12}$ and $|A \cap B| = 2^{11}$. Thus, $|A \cap B| = 2^{13} + 2^{12} - 2^{11} = 2^{11}(4 + 2 - 1) = 5 \cdot 2^{11} = 10240$.

(c) contain exactly four 1's?

Note that once we have chosen the 4 places to place the 1's, then we have to place 0's in the remaining 10 positions. Thus, we only have to count the number of ways to choose the 4 positions to place the 1's from all 14 positions. Hence there are $C(14, 4) = 1001$ bit strings containing exactly four 1's.

(d) contain at most twelve 1's?

Let us define the following sets:

$A = \{s|s$ is a bit string of length 14 which contains at most twelve 1's $\}$,
$B = \{s|s$ is a bit string of length 14 which contains exactly thirteen 1's $\}$, and
$C = \{s|s$ is a bit string of length 14 which contains exactly fourteen 1's $\}$.

Note that the intersection of any of the two sets defined above is the empty set, and $A \cup B \cup C$ is the set of bit string of length 14 and A is the set whose elements we need to count. Hence $|A \cup B \cup C| = |A| + |B| + |C| = 2^{14}$ where $|B| = C(14, 13) = 14$ and $|C| = C(14, 14) = 1$. Thus, $|A| = 2^{14} - 14 - 1 = 16369$.

(e) contain at least ten 1's.

Let $A_i = \{s|s$ is a bit string of length 14 which contains exactly i 1's $\}$ for $i = 10, 11, 12, 13, 14$.

Then, the number of bit strings containing at least ten 1's is $\sum_{i=10}^{14} |A_i| = C(14, 10) + C(14, 11) + C(14, 12) + C(14, 13) + C(14, 14) = 1001 + 364 + 91 + 14 + 1 = 1471$.

(f) have the sum of bits equal to 12.

This condition means there are exactly 12 1's (and 2 0's), so the total numbers of such strings is $C(14, 12) = 91$.

4. Let function f map the bit strings of length 4 onto the set $\{0, 1, 2, 3, 4\}$ where f assigns to each bit string of length 4 the number of 0 bits that the input string contains. For example: $f(0100) = 3$ and $f(0110) = 2$. Justify your answers.

(a) Is f one-to-one?

This function is not one-to-one, since $f(1110) = f(1011) = 1$, but $1110 \neq 1011$.

(b) Is f onto?

Since $f(0000) = 4, f(1000) = 3, f(1100) = 2, f(1011) = 1$ and $f(1111) = 0$, the range of f is $\{0, 1, 2, 3, 4\}$, which is the same as the codomain. Thus, f is onto.

5. Find the number of

(a) functions from a set with 5 elements to a set with 6 elements.

For each of the 5 inputs we can assign the outputs in 6 possible ways independently. Thus, there are $6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5 = 7776$ functions. This is an ordered selection with repetition.

(b) injective functions from a set with 5 elements to a set with 6 elements.

If a function is injective and an output is assigned to an input, then that output can not be reassigned to another input. Thus, there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = P(6, 5) = 720$ injective functions. This is an ordered selection without repetition.

(c) injective functions from a set with 6 elements to a set with 5 elements.

There are no such functions. It is not possible to choose 6 different outputs from 5 outputs. If a function is injective on a domain with 6 elements, then the codomain must have at least 6 elements.

(d) surjective functions from a set with 6 elements to a set with 6 elements.

A function from a set with 6 elements to a set with 6 elements is surjective if and only if it is injective. Thus, the number of surjective functions from a set with 6 elements to a set with 6 elements equals the number of injective functions from a set with 6 elements to a set with 6 elements and there are $6! = 720$ such functions.

6. From the word "ALGORITHM", how many ways can a

(a) three-letter word be selected?

Note the letters in the word ALGORITHM are all different. We have to choose three out of nine letters without repetition where the order is relevant. This is a permutation of 9 object taken 3 at a time, and hence, there are $P(9, 3) = 504$ choices.

(b) six-letter word be selected assuming the first letter must be a T?

Since the first letter is fixed, we have to choose five out of eight letters without repetition where the order is relevant. Hence there are $P(8, 5) = 6720$ choices.

(c) six-letter word be selected assuming the word contains the letter T?

Since the letter T can be placed in 6 different positions, there are $6 \cdot P(8, 5) = 40320$ choices.

(d) six-letter word be selected assuming the letters I and M have to be together in the word, either as IM or MI?

Note that the number of six-letter words containing IM is the same as the number of six-letter words containing MI. So, it is sufficient to calculation one of them and multiply the answer by two.
Think of IM as one letter. The other seven letters are A,L,G,O,R,T and H. Then there are 8 letters and we have to count the number of five-letter words containing the letter IM. Since the letter IM can be in

5 different positions, there are $5 \cdot P(7,4)$ five-letter words containing the letter IM which is the same as the number of six-letter words containing IM glued together.

Similarly, there are $5 \cdot P(7,4)$ different six-letter words containing MI glued together. Thus, by the Addition Principle, the total number of such words is $10 \cdot P(7,4) = 8400$.

Repetitions are not allowed in any of the selections, i.e., one letter can be selected only once.

7. How may hexadecimal strings of length 4

   (a) do not have any repeated digits?

   There are 16 hexadecimal digits. To find the number of hexadecimal strings of length 4 with no repeated digits, we select 4 digits out of 16 digits without repetition where the order is relevant. Thus, there are $P(16,4) = 43680$ such hexadecimal strings.

   (b) have at least one repeated digit?

   The total number of hexadecimal strings of length 4 is 16^4. Thus, the number of hexadecimal strings of length 4 with at least one repeated digit is $16^4 - P(16,4) = 21856$.

8. A computer programming team has 13 members, seven computer scientists and six software engineers.

   (a) How many groups of seven can be chosen to work on a project?

   We select 7 people out of 13 where the order of the selection does not matter. This is an unordered selection of 7 objects out of 13 objects without repetition, i.e., the combination of 13 objects taken 7 at a time. Thus, there are $C(13,7) = 1716$ such groups.

   (b) How many groups of seven can be chosen to work on a project if there must be three computer scientists and four software engineers in the group?

   We select 3 computer scientists out of 7 and 4 software engineers out of 6 where the order of the people within a group is not relevant. Thus, by the Multiplication Principle, there are $C(7,3) \cdot C(6,4) = 35 \cdot 15 = 525$ possibilities to create such groups.

   (c) Suppose that there are two members of the team who refuse to work together. How many groups of seven can be chosen to work on a project making sure these two people are never in the same group?

   Let us assume that person A and person B refuse to work together.

   The number of groups of seven having person A but not person B is $C(11,6) = 462$, similarly
   the number of groups of seven having person B but not person A is $C(11,6) = 462$, and
   the number of groups of seven having neither person A nor person B is $C(11,7) = 330$.

   The above mutually exclusive cases exhaust all the possibilities for making sure these two people are never in the same group of seven. Thus, according to the Addition Principle, there are $C(11,6) + C(11,6) + C(11,7) = 1254$ possible ways to select the groups.

9. There are ten points in plane, $P_1, P_2, P_3, \ldots P_{10}$, such that no three of them are colinear.

(a) How many distinct lines are determined by these ten points?

Since there are no three points that are on the same line, any two points selected from the given ten points determine a line uniquely. Thus, there are $C(10, 2) = 45$ lines can be created this way.

(b) How many triangles are formed by these ten points?

Since there are no three points that are on the same line, any three points selected from these ten points determine a triangle uniquely, and hence, the number triangles that can be formed is $C(10, 3) = 120$.

10. Three officers - a president, a treasurer, and a secretary - are to be chosen from among four people: Ann, Bob, Chandra and Diego.

(a) How many ways can a president, a treasurer and a secretary be selected from these four people?

There are 4 ways to pick the president, once the president is selected, there are 3 ways to pick the treasurer, and after the president and the treasurer is selected, the secretary can be chosen 2 ways. Hence, there are $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$ possible ways to choose a president, a treasurer, and a secretary from this group of four people.
Note the we could also think of selecting three out of four people, in order (as the positions are different), hence we have $P(4, 3) = 24$ different selections.

(b) Suppose that Bob is not qualified to be treasurer and Diego's other commitments make it impossible for him to be secretary. How many ways can a president, a treasurer and a secretary be chosen from the given group of four people?

We are going to consider the following three mutually exclusive alternatives:

Case 1: Bob is the president. Then, there are 2 ways to pick the secretary, and after the secretary is selected there are 2 ways to select the treasurer. In this scenario, there are $2 \cdot 2 = 4$ possibilities listed below:

President	Treasurer	Secretary
Bob	Diego	Ann
Bob	Chandra	Ann
Bob	Diego	Chandra
Bob	Ann	Chandra

Case 2: Bob is the secretary. Then, there is no restriction, there are 3 ways to pick the president, and 2 ways to pick the treasurer independently which gives the $6 = 3 \cdot 2$ possibilities listed below:

President	Treasurer	Secretary
Ann	Diego	Bob
Chandra	Diego	Bob
Diego	Ann	Bob
Chandra	Ann	Bob
Ann	Chandra	Bob
Diego	Chandra	Bob

Case 3: Bob is neither the president nor the secretary. Then there are two ways to pick the secretary since Diego can't be selected for this position, 2 ways to select the president, and 1 way to select the treasurer which gives the following $4 = 2 \cdot 2$ possibilities:

President	Treasurer	Secretary
Chandra	Diego	Ann
Diego	Chandra	Ann
Ann	Diego	Chandra
Diego	Ann	Chandra

According to the Addition Principle, altogether $4 + 6 + 4 = 14$ possible ways to select 3 people from a group of 4 considering the restrictions.

11. True or False? Justify your answer.

    (a) If there are 22 people in a room, then it is certain that at least 4 of them were born on the same day of the week.

    True. There are 22 people and seven days in a week. Since $22 = 3 \cdot 7 + 1$, by the Pigeonhole Principle, there are at least $3 + 1 = 4$ people in the group of 22 to be born the same day of the week. Notice that you get the same answer with the ceiling formula $\left\lceil \frac{22}{7} \right\rceil = 4$.

    (b) In a group of 700 people at least 2 have the same first and last initials.

    True. There are $26 \cdot 26 = 676$ possibilities for first/last name initials and 700 people. Then, by the Pigeonhole Principle, we conclude that there are at least $\left\lceil \frac{700}{676} \right\rceil = 2$ people in the group that have the same first and last initials.

12. In a class of 3000 students, the professor gives a multiple choice quiz with 4 questions with possible answers A, B, C, D or E. Each student answers each question. Among the 3000 students, at least how many students will turn in the exact same answers for the quiz questions?

    Let us calculate first how many different answers there are to this quiz. By the multiplication rule, we have $5^4 = 625$ different ways to answer the quiz. Since there are 3000 students, and $3000 = 4 \cdot 625 + 500$, there are at least $4 + 1 = 5$ students who gave the very same answers by the Pigeonhole Principle. Notice that we get the same answer with the ceiling function formula $\left\lceil \frac{3000}{625} \right\rceil = 5$.

13. What is the minimum number of people you need to guarantee that there are three among them who were born on the same day? Recall there are 366 possible birthdays.

    By the Pigeonhole Principle we have to find the minimum number of people, N, such that $\left\lceil \frac{N}{366} \right\rceil = 3$. There should be at least $N = 366 \cdot 2 + 1 = 733$ people to guarantee that at least three have the same birthday.

14. There are 193 member states of the United Nations. If you have 195 diplomats from these states in a room, then which of the following statements can be inferred?

    1. It is certain that at least 2 of them are from the same state. Correct.

    2. It is certain that at least 3 of them are from the same state.

    By the Pigeonhole Principle we can infer statement 1, since $\left\lceil \frac{195}{193} \right\rceil = 2$.

    We need at least $193 \cdot 2 + 1 = 387$ in order to be able to infer statement 2.

15. A device produces random 64-bit integers at a rate of one billion per second. After how many years of running is it unavoidable that the device produces an output for the second time? Round to the nearest number of years.

    There are 2^{64} different 64-bit integers. We need to have at least $2^{64} + 1$ 64-bit integers so that we are certain of a repetition. Thus, after $\frac{2^{64} + 1}{10^9 \cdot 60 \cdot 60 \cdot 24 \cdot 365.25} \approx 585$ years of running the device it is unavoidable that the device produces an output for the second time.

16. A penny collection contains twelve pennies from the year 1967, seven pennies from the year 1968, and eleven pennies from the year 1971. If you were to pick some pennies without looking at the dates, how many must you pick to be sure of getting at least five pennies from the same year?

    There are 3 different types of pennies which play the roles of the "boxes" in the Pigeonhole Principle. Thus, we need to find the minimum number pennies, N, such that $\left\lceil \frac{N}{3} \right\rceil = 5$. This means picking $N = 4 \cdot 3 + 1 = 13$ pennies guarantee that at least 5 pennies are from the same year.

17. A bag contains 100 apples, 100 bananas, 100 oranges and 100 pears. If someone randomly picks a fruit from the bag every second,

    (a) how long will it take to be sure to have at least a dozen of the same kind of fruit?

    There are 4 different types of fruit. By the Pigeonhole Principle we have to find the minimum number fruit, N, such that $\left\lceil \frac{N}{4} \right\rceil = 12$. Thus, $N = 4 \cdot 11 + 1 = 45$ fruit guarantee that we have at least a dozen of the same kind of fruit. Since we pick a fruit from the bag in every second, we need minimum 45 seconds to pick fruits to ensure we have at least a dozen of the same kind.

    (b) how long will it take to be sure to have at least a dozen bananas?

    In the worst case scenario, all the apples, pears, and oranges are picked before we have a chance to pick 12 bananas. Thus, the selection of $300 + 12 = 312$ fruit guarantees that at least a dozen of them are bananas, and hence it takes at least 312 seconds, or 5.2 minutes. The selection of 311 bananas does not suffice in the worst case scenario.

    Note that the Pigeonhole Principle can not be applied here. The Pigeonhole Principle does not guarantee repetition of a specific fruit such as a banana, it only guarantees the repetition for some kind of fruit (apples, pears, oranges, bananas) but we don't know which kind.

18. A restaurant serves a 3-course lunch option for tourists. The tourists could choose between the following two options:

    Option 1: a soup, a main course and a dessert,

    Option 2: an appetizer, a soup and a main course.

    There are 4 options for appetizers, 5 options for soup, 10 options for the main course and 3 different strudel options for dessert.

    What is the smallest number of tourists that have to visit this restaurant to guarantee that at least 10 tourists ordered the same 3-course lunch option?

    This problem requires you to use the Multiplication, Addition and the Pigeonhole Principles. There are $5 \cdot 10 \cdot 3 + 4 \cdot 5 \cdot 10 = 350$ possible 3-course meals that are offered for the tourists. According to the Pigeonhole Principle, at least $9 \cdot 350 + 1 = 3151$ tourists have to visit to guarantee that at least 10 of them ordered the same 3-course meal.

19. An ice cream store gives the following options for the customers to create their own ice cream sundaes:

    1. Pick 3 different (can't select a flavour twice) ice cream flavours out of 8.

2. Pick 2 different (no repetition) toppings out of 5.

How many times does Helena have to visit this ice cream store so that it is unavoidable that she would eat the same sundae for the third time?

This problem requires you to use combinations, the Multiplication and the Pigeonhole Principles. There are $C(8,3) \cdot C(5,2) = 56 \cdot 10 = 560$ different types of sundaes, since the order of the selection neither of the flavours nor the toppings are matter. According to the Pigeonhole Principle, Helena must visit the ice cream store at least $2 \cdot 560 + 1 = 1121$ times until it is unavoidable that she eats the same sundae for the third time.

20. A password generator creates passwords which consist of a minimum of 6 and a maximum of 8 characters. The generator can use either digits (0-9), lower case letters, upper case letters, or the characters !? (exclamation and question marks). How many passwords need to be generated until it is unavoidable that the same password is created for the second time? Leave your answer in unevaluated form.

    This problem requires you to use the Multiplication, Addition and the Pigeonhole Principles. There are $(10+52+2)^6 + (10+52+2)^7 + (10+52+2)^8$ different passwords. By the Pigeonhole Principle $(10+52+2)^6 + (10+52+2)^7 + (10+52+2)^8 + 1$ must be generated until it is unavoidable that a password is created for the second time.

21. **Counting with Recursion.** Let a_n be the number of bit strings of length n which contain exactly two 0's.

    (a) Find a_1.

    There is no bit string of length 1 which contain exactly two 0's, so $a_1 = 0$.

    (b) Find the recursive formula for $a_n, n \geq 2$.

    For $n \geq 2$, any such bit string c of length n is generated recursively by one of the following mutually exclusive alternatives:

    Case 1: $c = 1b$, where b is a bit string of length $n-1$ which contains exactly two 0's. There are a_{n-1} such bit strings.
    Case 2: $c = 0b$, where b is a bit string of length $n-1$ which contains exactly one 0's. There are $n-1$ such bit strings.

    Case 1 and 2 are mutually exclusive, which means we don't generate bit strings multiple times. Therefore $a_n = a_{n-1} + (n-1), n \geq 2$, which is a first degree, linear non-homogeneous recurrence relation. Note that we used the Addition Principle and recursion to find $a_n, n \geq 2$.

    (c) Use elementary counting method to find a closed-form formula for a_n.

    Once we chose the 2 places to place the 0's, then we have to place 1's for the remaining $n-2$ positions. Thus, we have to count the number of ways to choose the 2 positions to place the 0's from all n positions. Hence there are $C(n,2) = \dfrac{n(n-1)}{2}$ bit strings containing exactly two 0's.

    Note that it is not always possible or convenient to use elementary counting methods. Frequently, recursive counting methods, which divide the problem into smaller sub-problems, are used in complexity analysis.

    (d) Use induction to show that $\{a_n\}$ defined recursively by $a_n = a_{n-1} + (n-1)$, $n \geq 2$, with $a_1 = 0$ is equivalent to the closed formula $a_n = \dfrac{n(n-1)}{2}$ for all positive integers n.

Basis Step: The $n = 1$ case is true, since $a_1 = 0$ by definition and $\frac{n(n-1)}{2} = 0$.

Inductive Step: Assume that $a_n = \frac{n(n-1)}{2}$ for an arbitrary positive integer n.

We wish to prove that $a_{n+1} = \frac{n(n+1)}{2}$.

Using the inductive hypothesis and the recursive definition,
$a_{n+1} = a_n + n = \frac{n(n-1)}{2} + n = \frac{n(n-1)}{2} + \frac{2n}{2} = \frac{n(n+1)}{2}$ which shows that a_{n+1} has the required form.

The proof by induction is complete.

22. **Counting in Complexity Analysis.** Counting has many applications in computer science. We use systematic counting techniques to count the number of operations used by an algorithm to analyze time complexity. This problem asks you analyze a simple Python code segment.

The range() function returns a list of integers, starting from 0 and increments by 1 and ends at a certain number. See the following examples:

range(7) returns the list of integers 0, 1, 2, 3, 4, 5, 6 (not including 7).

range(4,8) returns the list 4, 5, 6, 7 (not including 8).

(a) Count the number of additions and multiplications in the following Python code segment:

```
for i in range(26):
    for j in range(34,76):
        k=i+j+3
        l=2*i+3*j
        m=i*j+1
print(2*k+l+m)
```

The outer loop iterates 26 times. For each iteration of the outer loop, the inner loop iterates $76 - 34 = 42$ times. That is a total of $26 \cdot 42$ iterations all together.
For each iteration of the nested for loops, there are 3 sequential statements inside containing 4 additions and 3 multiplications.
After the nested for loops are completed, the print statement contains 2 additions and one multiplication.
Altogether, there are $26 \cdot 42 \cdot (3 + 4) + 3 = 7647$ additions and multiplications in this code segment.
Notice how the multiplication and addition principles were applied in this simple complexity analysis.

(b) Count the number of additions and multiplications in the following Python code segment:

```
value=0
for x in range(50):
    for y in range(57):
        if y>x:
            value=value+y
print(value)
```

When $x = 0$, then the y-loop runs 56 times.
When $x = 1$, then the y-loop runs 55 times.

When $x = 2$, then the y-loop runs 54 times.
etc
When $x = 49$, then the y-loop runs 7 times.

One addition is performed for each iteration.

Thus, the number additions performed in this program segment is:
$56 + 55+, ..., +7$ which is an arithmetic sequence with difference 1 and with 50 terms.

To find the sum we use the formula for the sum of the terms of an arithmetic sequence:
$\dfrac{(7+56)\cdot 50}{2} = 1575.$